AN INTRODUCTION TO DIGITAL IMAGE CORRELATION

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Part 2/2

Recap

- Corrected deformed image $\tilde{g}(\underline{\mathbf{x}})$

$$\tilde{g}_{\underline{\mathbf{v}}}(\underline{\mathbf{x}}) = g(\underline{\mathbf{x}} + \underline{\mathbf{v}}(\underline{\mathbf{x}}))$$

- DIC is based on minimization of

$$\mathrm{T}[\underline{\mathbf{v}}] = \left\| \tilde{g}_{\underline{\mathbf{v}}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}}) \right\|^2$$

$$\underline{\mathbf{u}} = \operatorname{argmin} \mathrm{T}[\underline{\mathbf{v}}]$$

Relaxation

Let us assume that the displacement $\{v^{(n)}\}$ is close to the solution

$$\mathrm{T}(\{\mathbf{v}+\delta\mathbf{v}\}) = \sum_{\underline{\mathbf{x}}} \left(\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}} + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}})) - f(\underline{\mathbf{x}})\right)^2$$

$$\approx \sum_{\underline{\mathbf{x}}} \left(\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}})\right)^2$$

The latter is a mere quadratic form easy to minimize

Relaxation

- Select index i

$$\frac{\partial \mathrm{T}}{\partial \delta v_i} = \frac{\partial}{\partial \delta v_i} \sum_{\underline{\mathbf{x}}} \left(\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) + \delta v_j \, \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}}) \right)^2$$

Relaxation

- Select index *i*

$$\frac{\partial \mathrm{T}}{\partial \delta v_i} = \frac{\partial}{\partial \delta v_i} \sum_{\underline{\mathbf{x}}} \left(\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}}) \right)^2$$

Relaxation

- Select index i

$$\frac{\partial \mathrm{T}}{\partial \delta v_i} = \frac{\partial}{\partial \delta v_i} \sum_{\underline{\mathbf{x}}} \left(\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}}) \right)^2$$

$$= \sum_{\underline{\mathbf{x}}} \frac{\partial}{\partial \delta v_i} \left(\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}}) \right)^2$$

Relaxation

- Select index i

$$\frac{\partial \mathrm{T}}{\partial \delta v_i} = \frac{\partial}{\partial \delta v_i} \sum_{\underline{\boldsymbol{x}}} \left(\tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\boldsymbol{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) - f(\underline{\boldsymbol{x}}) \right)^2$$

$$= \sum_{\underline{\boldsymbol{x}}} \frac{\partial}{\partial \delta v_i} \left(\tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\boldsymbol{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) - f(\underline{\boldsymbol{x}}) \right)^2$$

$$= \sum_{\underline{\boldsymbol{x}}} 2 \left(\tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\boldsymbol{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) - f(\underline{\boldsymbol{x}}) \right) \frac{\partial}{\partial \delta v_i} \left(\tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) + \delta v_k \underline{\boldsymbol{\psi}}_k(\underline{\boldsymbol{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) - f(\underline{\boldsymbol{x}}) \right)$$

Relaxation

- Select index i

$$\frac{\partial \mathrm{T}}{\partial \delta v_i} = \frac{\partial}{\partial \delta v_i} \sum_{\underline{\boldsymbol{x}}} \left(\tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\boldsymbol{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) - f(\underline{\boldsymbol{x}}) \right)^2$$

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$$= \sum_{\underline{\boldsymbol{x}}} 2 \left(\tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\boldsymbol{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) - f(\underline{\boldsymbol{x}}) \right) \frac{\partial}{\partial \delta v_i} \left(\tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) + \delta v_k \underline{\boldsymbol{\psi}}_k(\underline{\boldsymbol{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\boldsymbol{v}\}}(\underline{\boldsymbol{x}}) - f(\underline{\boldsymbol{x}}) \right)$$

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Relaxation

- Select index i

$$\frac{\partial \mathrm{T}}{\partial \delta v_i} = \frac{\partial}{\partial \delta v_i} \sum_{\underline{x}} \left(\tilde{g}_{\{v\}}(\underline{x}) + \delta v_j \underline{\psi}_j(\underline{x}) \cdot \underline{\nabla} \tilde{g}_{\{v\}}(\underline{x}) - f(\underline{x}) \right)^2$$

$$= \sum_{\underline{x}} \frac{\partial}{\partial \delta v_i} \left(\tilde{g}_{\{v\}}(\underline{x}) + \delta v_j \underline{\psi}_j(\underline{x}) \cdot \underline{\nabla} \tilde{g}_{\{v\}}(\underline{x}) - f(\underline{x}) \right)^2$$

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$$= \sum_{\underline{x}} 2 \left(\tilde{g}_{\{v\}}(\underline{x}) + \delta v_j \underline{\psi}_j(\underline{x}) \cdot \underline{\nabla} \tilde{g}_{\{v\}}(\underline{x}) - f(\underline{x}) \right) \left(\underline{\psi}_i(\underline{x}) \cdot \underline{\nabla} \tilde{g}_{\{v\}}(\underline{x}) \right)$$

$$= \sum_{\underline{x}} 2 \left(\delta v_j \underline{\psi}_j(\underline{x}) \cdot \underline{\nabla} \tilde{g}_{\{v\}}(\underline{x}) \right) \left(\underline{\psi}_i(\underline{x}) \cdot \underline{\nabla} \tilde{g}_{\{v\}}(\underline{x}) \right) - \sum_{\underline{x}} 2 \left(f(\underline{x}) - \tilde{g}_{\{v\}}(\underline{x}) \right) \left(\underline{\psi}_i(\underline{x}) \cdot \underline{\nabla} \tilde{g}_{\{v\}}(\underline{x}) \right)$$

=

=

Relaxation

- Select index i

$$\frac{\partial \mathrm{T}}{\partial \delta v_i} = \frac{\partial}{\partial \delta v_i} \sum_{\underline{\mathbf{x}}} \left(\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}}) \right)^2$$

$$= \sum_{\underline{\mathbf{x}}} \frac{\partial}{\partial \delta v_i} \left(\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}}) \right)^2$$

$$= \sum_{\underline{\mathbf{x}}} 2 \left(\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}}) \right) \frac{\partial}{\partial \delta v_i} \left(\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}}) \right)$$

$$= \sum_{\underline{\mathbf{x}}} 2 \left(\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}}) \right) \left(\underline{\boldsymbol{\psi}}_i(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) \right)$$

$$= \sum_{\underline{\mathbf{x}}} 2 \left(\delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) \right) \left(\underline{\boldsymbol{\psi}}_i(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) \right) - \sum_{\underline{\mathbf{x}}} 2 \left(f(\underline{\mathbf{x}}) - \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) \right) \left(\underline{\boldsymbol{\psi}}_i(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) \right)$$

$$= 2 \left[\sum_{\underline{\mathbf{x}}} \left(\underline{\boldsymbol{\psi}}_i(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) \right) \left(\underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) \right) \right] \delta v_j - 2 \sum_{\underline{\mathbf{x}}} \left(f(\underline{\mathbf{x}}) - \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) \right) \left(\underline{\boldsymbol{\psi}}_i(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) \right)$$

Relaxation

- Select index i

$$\frac{\partial \mathrm{T}}{\partial \delta v_i} = \frac{\partial}{\partial \delta v_i} \sum_{\underline{\mathbf{x}}} \left(\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}}) \right)^2$$

$$= \sum_{\underline{\mathbf{x}}} \frac{\partial}{\partial \delta v_i} \left(\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) + \delta v_j \underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}}) \right)^2$$

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$$= 2 \left[\sum_{\underline{\mathbf{x}}} \left(\underline{\boldsymbol{\psi}}_i(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) \right) \left(\underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) \right) \right] \delta v_j - 2 \sum_{\underline{\mathbf{x}}} \left(f(\underline{\mathbf{x}}) - \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) \right) \left(\underline{\boldsymbol{\psi}}_i(\underline{\mathbf{x}}) \cdot \underline{\nabla} \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}}) \right)$$

$$= 2 \left(M_{ij} \delta v_j - b_i \right)$$

Relaxation

$$\frac{\partial \mathrm{T}}{\partial\{\delta\mathbf{v}\}} = \{\mathbf{0}\}$$ leads to $$M_{ij}\delta v_j = b_i$$

with

$$M_{ij} = \sum_{\underline{\mathbf{x}}} \left(\underline{\boldsymbol{\psi}}_i(\underline{\mathbf{x}}) \cdot \underline{\nabla}\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}})\right)\left(\underline{\boldsymbol{\psi}}_j(\underline{\mathbf{x}}) \cdot \underline{\nabla}\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}})\right)$$

$$b_i = \sum_{\underline{\mathbf{x}}} \left(\underline{\boldsymbol{\psi}}_i(\underline{\mathbf{x}}) \cdot \underline{\nabla}\tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}})\right)\left(f(\underline{\mathbf{x}}) - \tilde{g}_{\{\mathbf{v}\}}(\underline{\mathbf{x}})\right)$$

Outline

- Uncertainty quantification
- 1D strain measurement
- Consequences for DIC
- Identification
- Examples
- Conclusions

UNCERTAINTY QUANTIFICATION

Measurements

- The **measurement result** consists of a set of **quantity values** with **units** accompanied by a **measurement uncertainty**

[99-12:2007, 2007, *VIM, International Organization for Standardization*]

Measurement Uncertainty

- **Measurement uncertainty**: parameter that characterizes the dispersion of the **quantity values** that are being attributed to a **measurand**, based on the information used
- **Standard measurement uncertainty**: measurement uncertainty expressed as a standard deviation

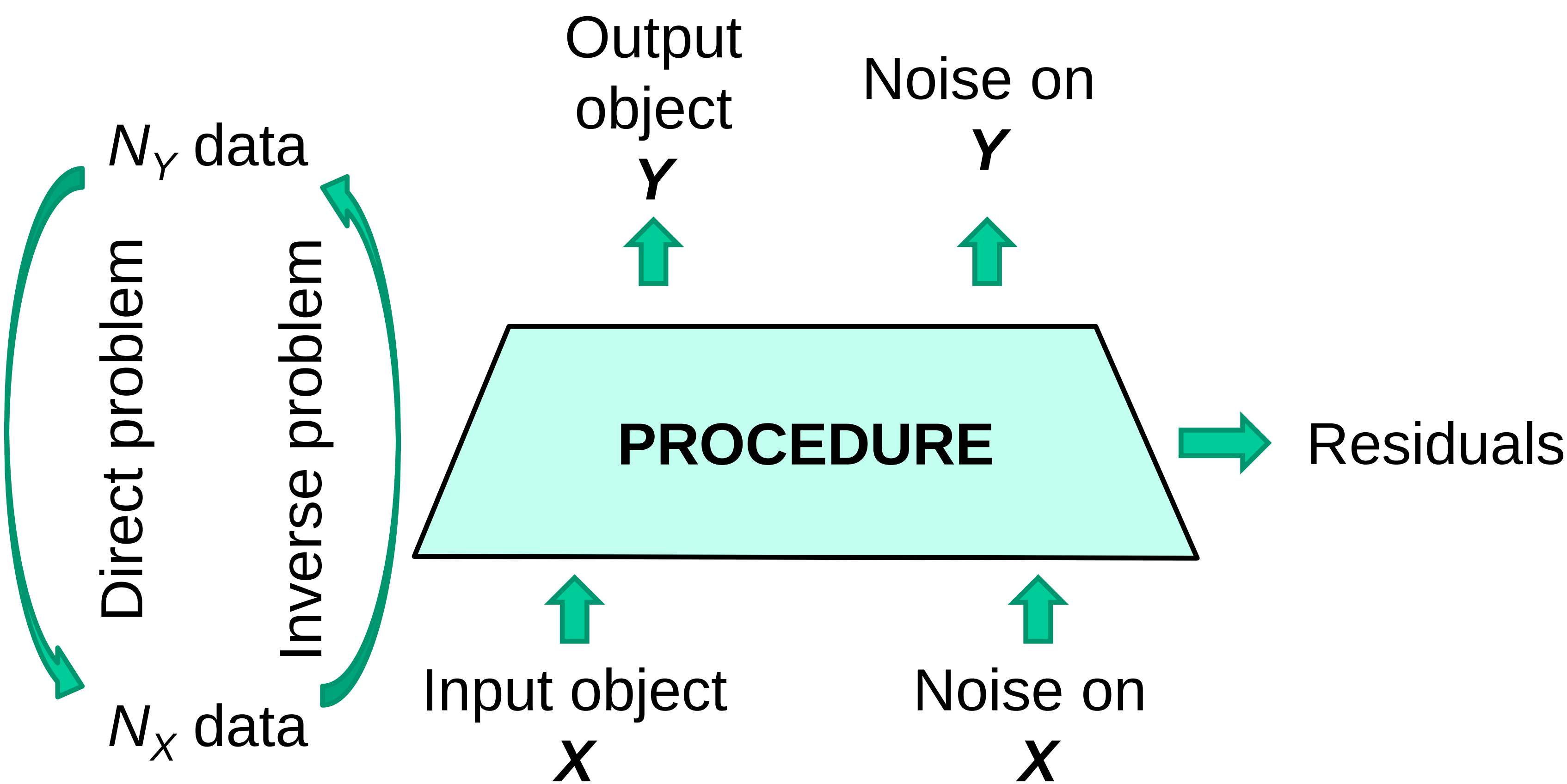
[99-12:2007, 2007, *VIM, International Organization for Standardization*]

Overall Philosophy

- When noise is known, the *uncertainty of the measurand* with a specific procedure can be characterized, thereby upgrading estimation to measurement

- One can further devise THE procedure that minimizes the resulting uncertainty: ***Optimal measurement***

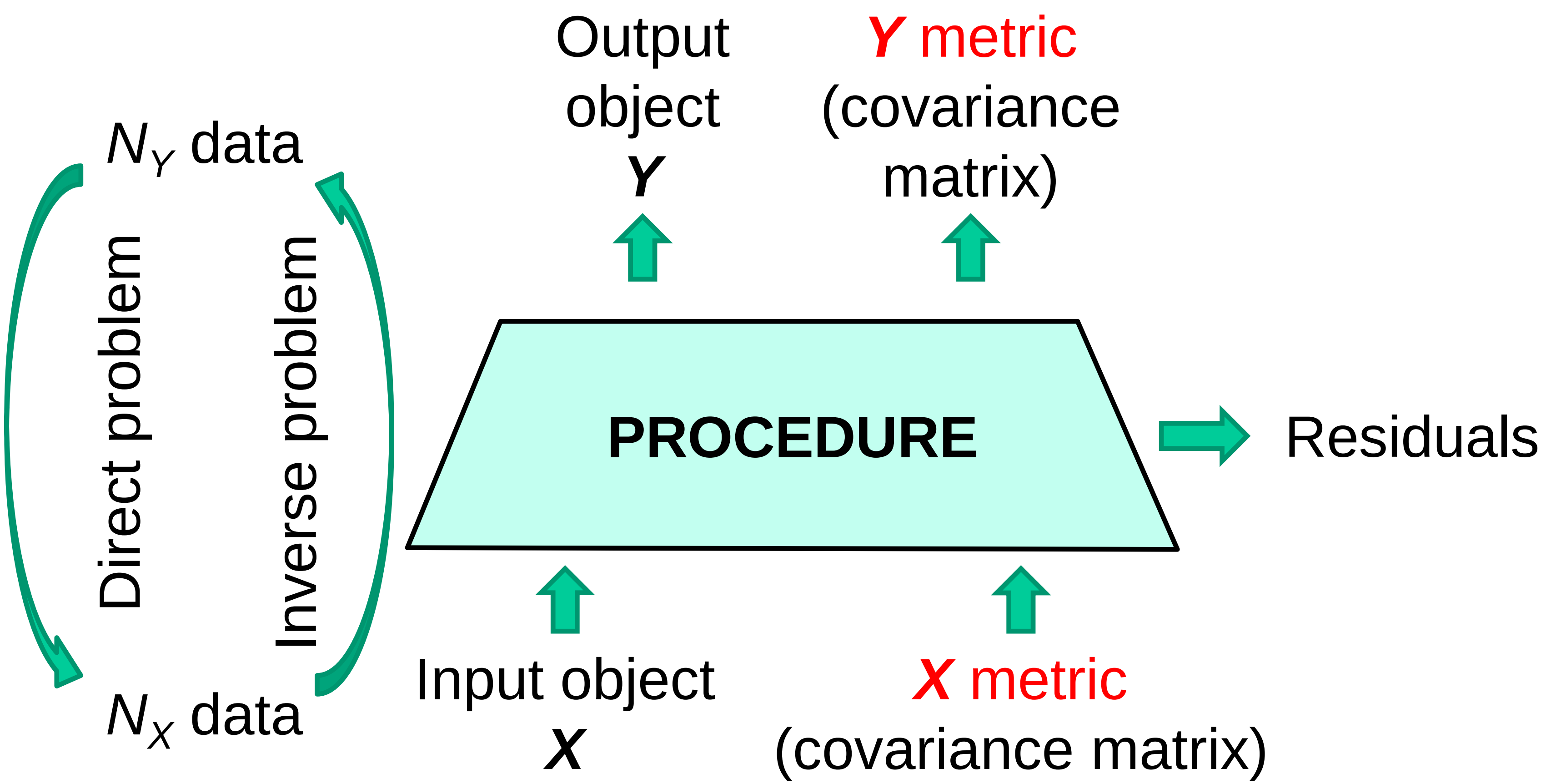
Generic Building Block



$$N_X >> N_Y$$

[Neggers *et al.*, 2018, *Arch. Comput. Mech. Eng.* 25(1) pp.143-164]

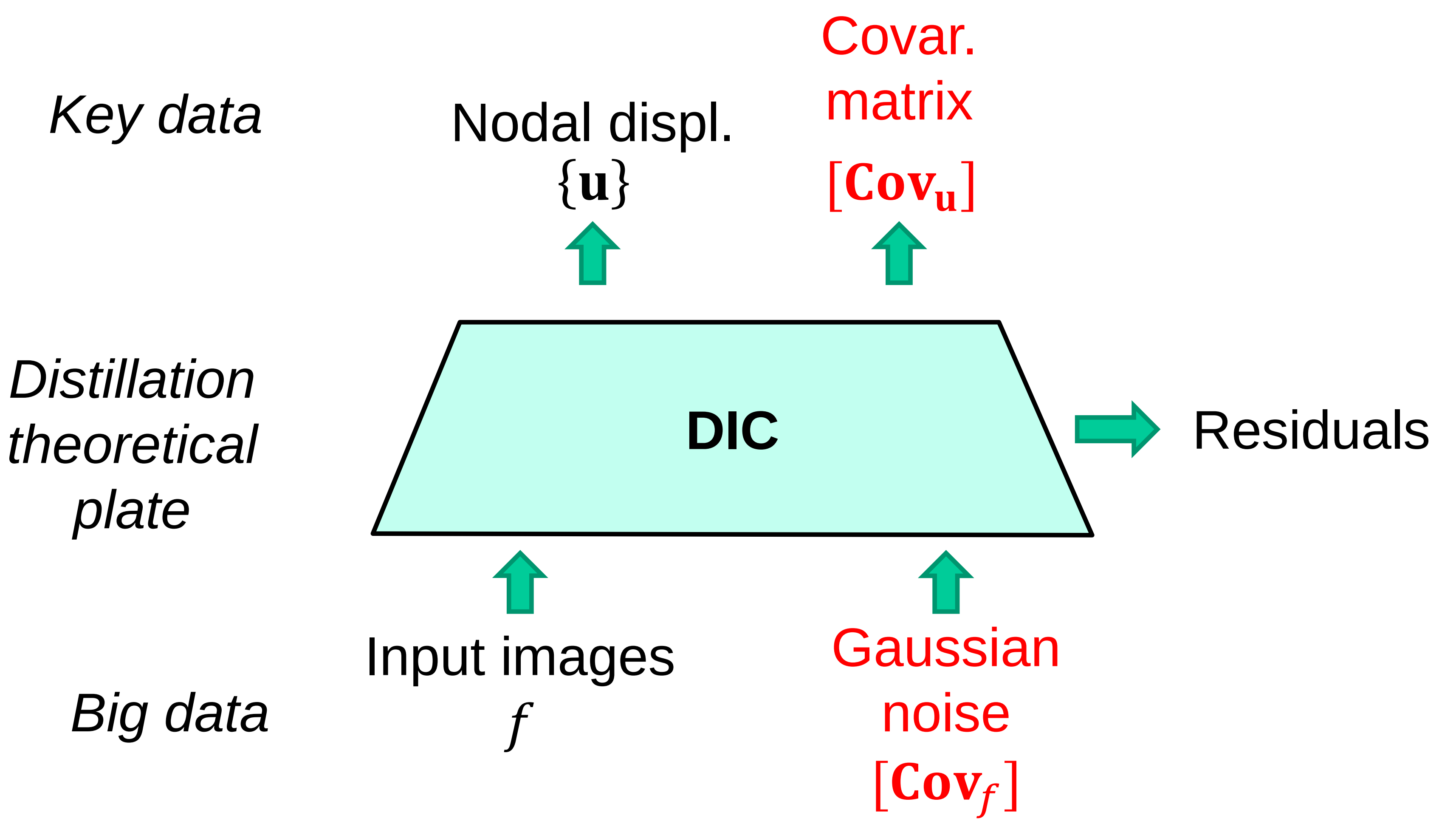
Generic Building Block



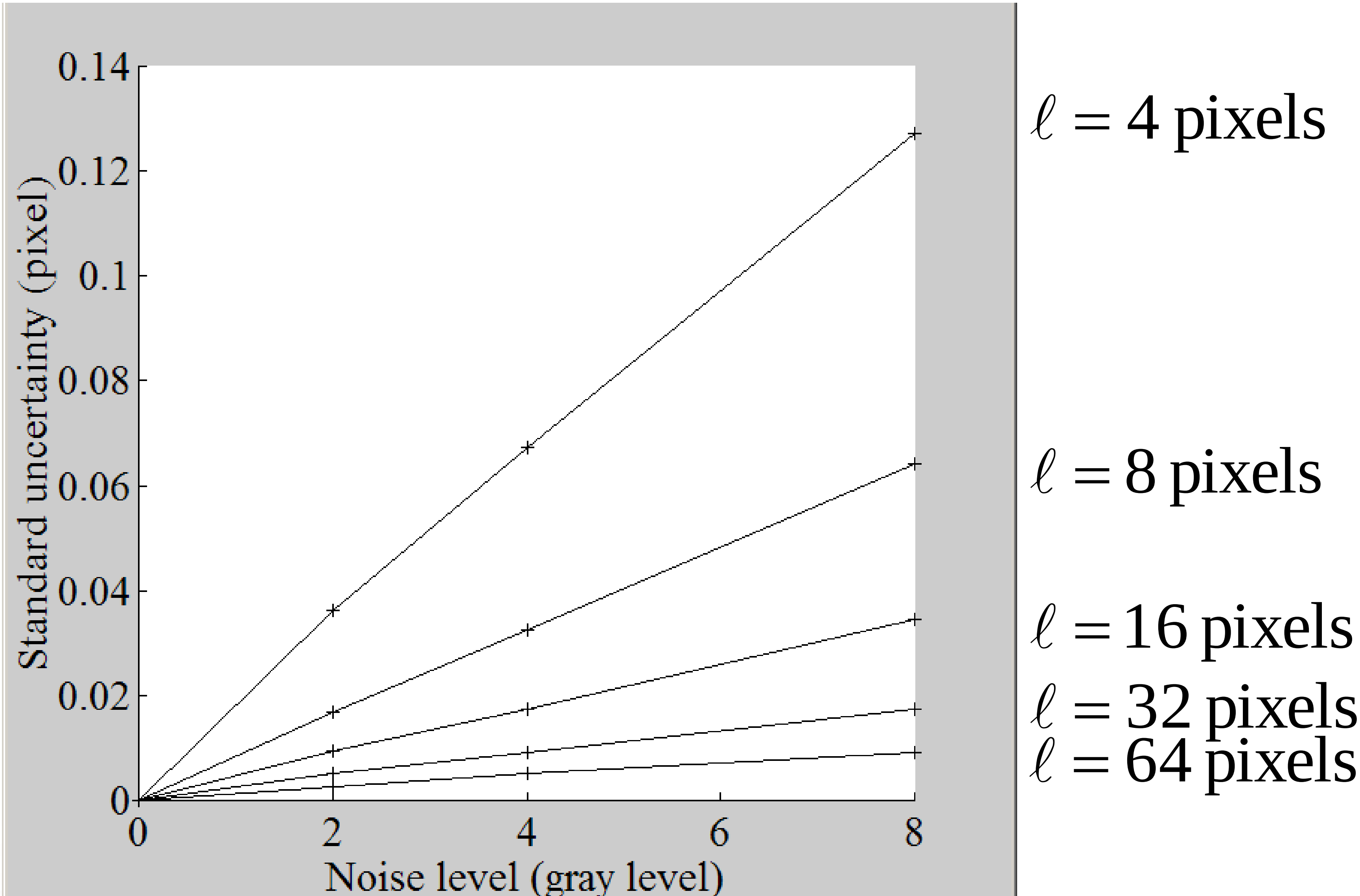
$N_X >> N_Y$

[Neggers *et al.*, 2018, *Arch. Comput. Mech. Eng.* 25(1) pp.143-164]

DIC/DVC

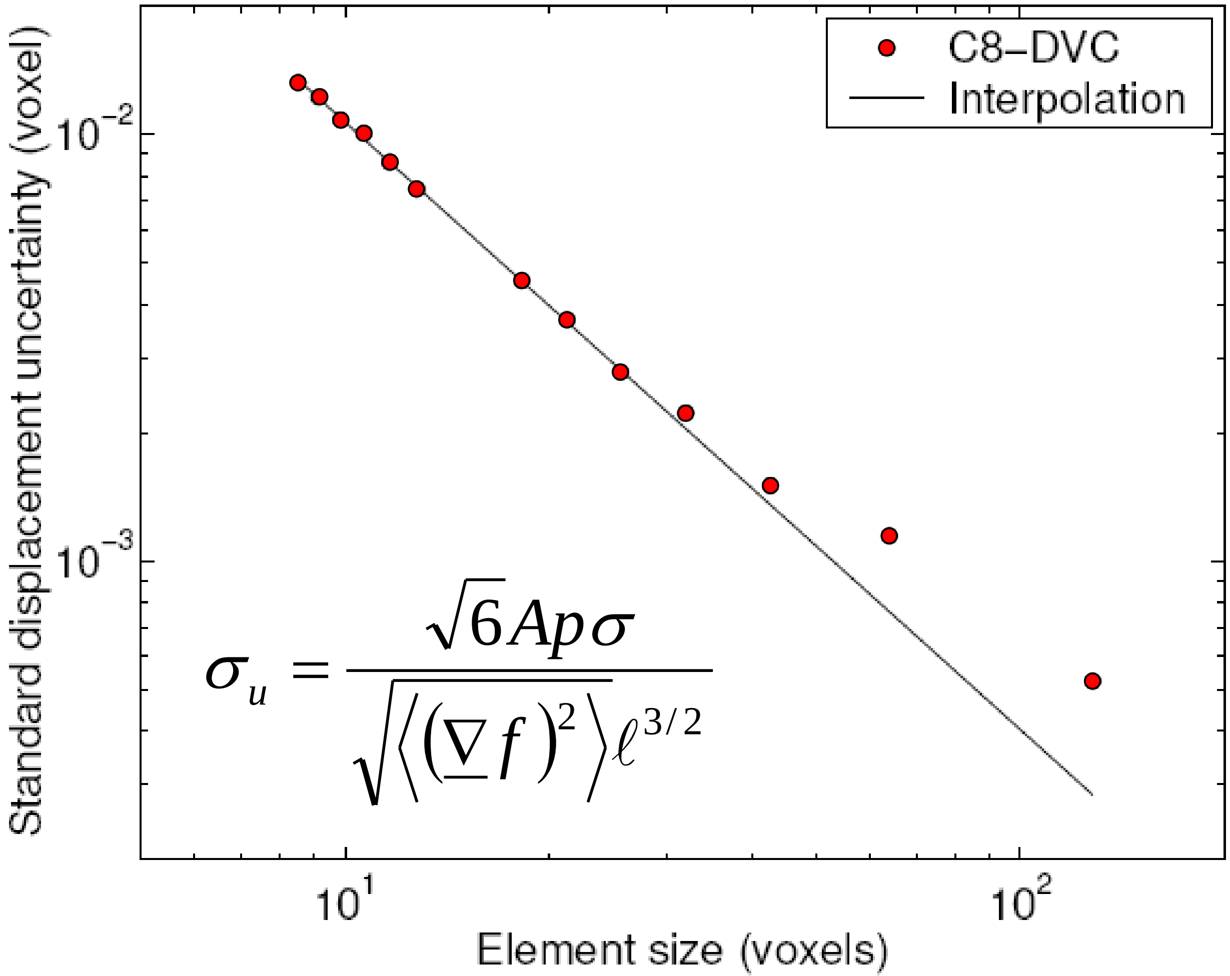

Key data
Distillation theoretical plate
Big data
Nodal displ.
$\{\mathbf{u}\}$
Covar. matrix
$[\mathbf{Cov_u}]$
DIC
Residuals
Input images
f
Gaussian noise
$[\mathbf{Cov}_f]$

Displacement Uncertainty vs. Noise level



Resolution Analysis*

"Heisenberg uncertainty"



$$\sigma_u = \frac{\sqrt{6}Ap\sigma}{\sqrt{\left\langle (\underline{\nabla} f)^2 \right\rangle}\ell^{3/2}}$$

*[Leclerc *et al.*, 2011, *Exp. Mech* 51(4) 479-490]

Noise

- Assume white noise, η, of variance $2\sigma^2$ affects the deformed image only

Why?

$$\langle \eta(\underline{\mathbf{x}}) \rangle = 0$$

$$\langle \eta(\underline{\mathbf{x}})\eta(\underline{\mathbf{x}}') \rangle = 2\sigma^2 \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}')$$

- No change in $[\mathbf{M}]$, change in $\{\mathbf{b}\}$

Noise

Noise produces change in estimation of displacement parameters $\{\delta \mathbf{u}\}$ such that

$$[\mathbf{M}]\{\delta \mathbf{u}\} = \{\delta \mathbf{b}\}$$

$$\delta b_i = \sum_{\underline{\mathbf{x}}} \left(\underline{\boldsymbol{\psi}}_i(\underline{\mathbf{x}}) \cdot \underline{\nabla} f(\underline{\mathbf{x}}) \right) \eta(\underline{\mathbf{x}})$$

Displacement Resolution

$$\{\delta\mathbf{u}\} = [\mathbf{M}]^{-1}\{\delta\mathbf{b}\}$$

Solution is unbiased

$$\langle\{\delta\mathbf{u}\}\rangle = [\mathbf{M}]^{-1}\langle\{\delta\mathbf{b}\}\rangle = \{\mathbf{0}\}$$

Displacement Resolution

$$\{\delta \mathbf{u}\} = [\mathbf{M}]^{-1}\{\delta \mathbf{b}\}$$

Covariance matrix

$$\langle \delta u_i \delta u_j \rangle = M_{ik}^{-1} M_{jl}^{-1} \langle \delta b_k \delta b_l \rangle$$

$$\langle \delta b_k \delta b_l \rangle = \sum_{\underline{\mathbf{x}}, \underline{\mathbf{x}}'} \left(\underline{\boldsymbol{\psi}}_k(\underline{\mathbf{x}}) \cdot \underline{\nabla} f(\underline{\mathbf{x}}) \right) \left(\underline{\boldsymbol{\psi}}_l(\underline{\mathbf{x}}') \cdot \underline{\nabla} f(\underline{\mathbf{x}}') \right) \langle \eta(\underline{\mathbf{x}}) \eta(\underline{\mathbf{x}}') \rangle$$

$$= 2\sigma^2 \sum_{\underline{\mathbf{x}}} \left(\underline{\boldsymbol{\psi}}_k(\underline{\mathbf{x}}) \cdot \underline{\nabla} f(\underline{\mathbf{x}}) \right) \left(\underline{\boldsymbol{\psi}}_l(\underline{\mathbf{x}}) \cdot \underline{\nabla} f(\underline{\mathbf{x}}) \right)$$

$$= 2\sigma^2 M_{kl}$$

Displacement Resolution

$$\{\delta \mathbf{u}\} = [\mathbf{M}]^{-1}\{\delta \mathbf{b}\}$$

Covariance matrix

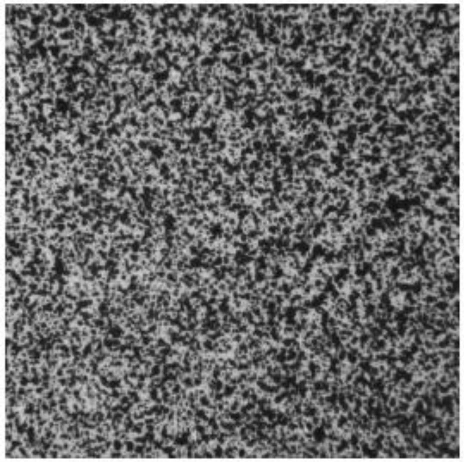
$$\left\langle \delta u_i \delta u_j \right\rangle = 2\sigma^2 M_{ik}^{-1} M_{kl} M_{jl}^{-1}$$

$$= 2\sigma^2 M_{ij}^{-1}$$

$$[\mathbf{Cov_u}] = \left\langle \{\delta \mathbf{u}\}\{\delta \mathbf{u}\}^{\mathrm{T}} \right\rangle = 2\sigma^2 [\mathbf{M}]^{-1}$$

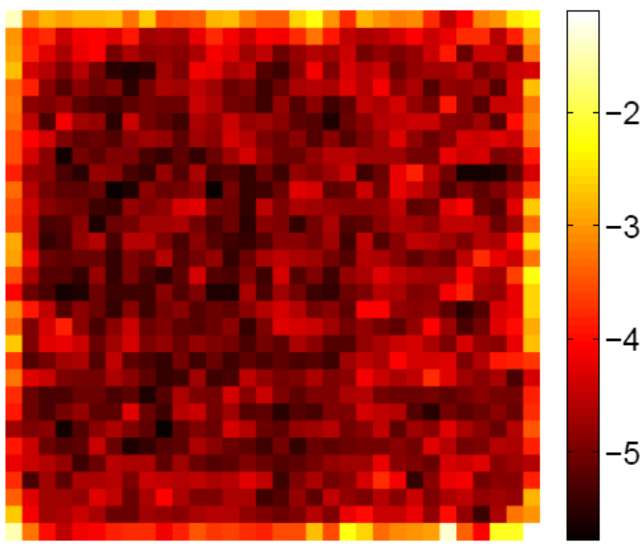
[SR *et al., 2006, Int. J. Fract.* 140 pp. 141-157]

Noise Sensitivity Analysis

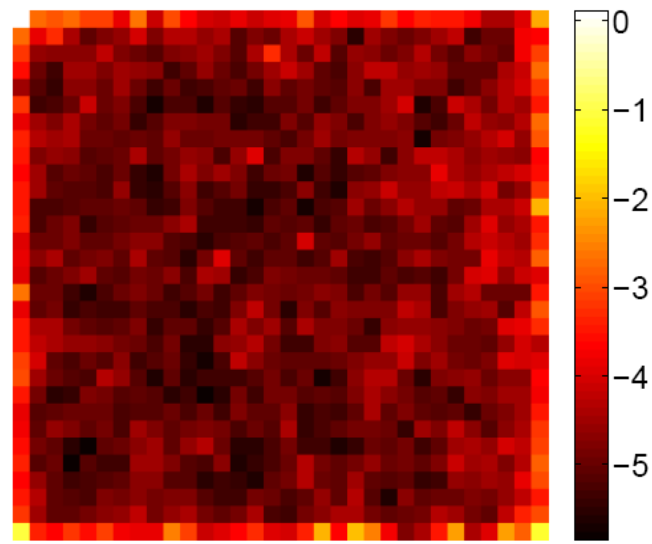


$$[\mathbf{M}]\{\delta\mathbf{u}\} = \{\delta\mathbf{b}\}$$

$$\left\langle \{\delta\mathbf{u}\}\{\delta\mathbf{u}\}^{\mathrm{T}} \right\rangle = 2\sigma^2[\mathbf{M}]^{-1}$$



$$\log\left\langle \{\delta u_x\}\{\delta u_x\}^{\mathrm{T}} \right\rangle_e$$



$$\log\left\langle \{\delta u_y\}\{\delta u_y\}^{\mathrm{T}} \right\rangle_e$$

1D STRAIN MEASUREMENT

Trivial Example

- 1D displacement measurements given at equispaced points

$$u_i = u(x_i)$$

- Model: strain is uniform + measurement is ‘decorated’ by **Gaussian white** noise

$$u_i = \varepsilon x_i + v + \eta_i$$

- Question: what is the strain ε?

Gaussian Noise?

- Means that η is normally distributed with 0 mean and variance σ^2
- Pdf $p(\eta)$

$$p(\eta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\eta^2}{2\sigma^2}\right)$$

$$\langle \eta \rangle = 0 \qquad \langle \eta^2 \rangle = \sigma^2$$

- Key property: stability upon addition (central limit theorem)

White Noise?

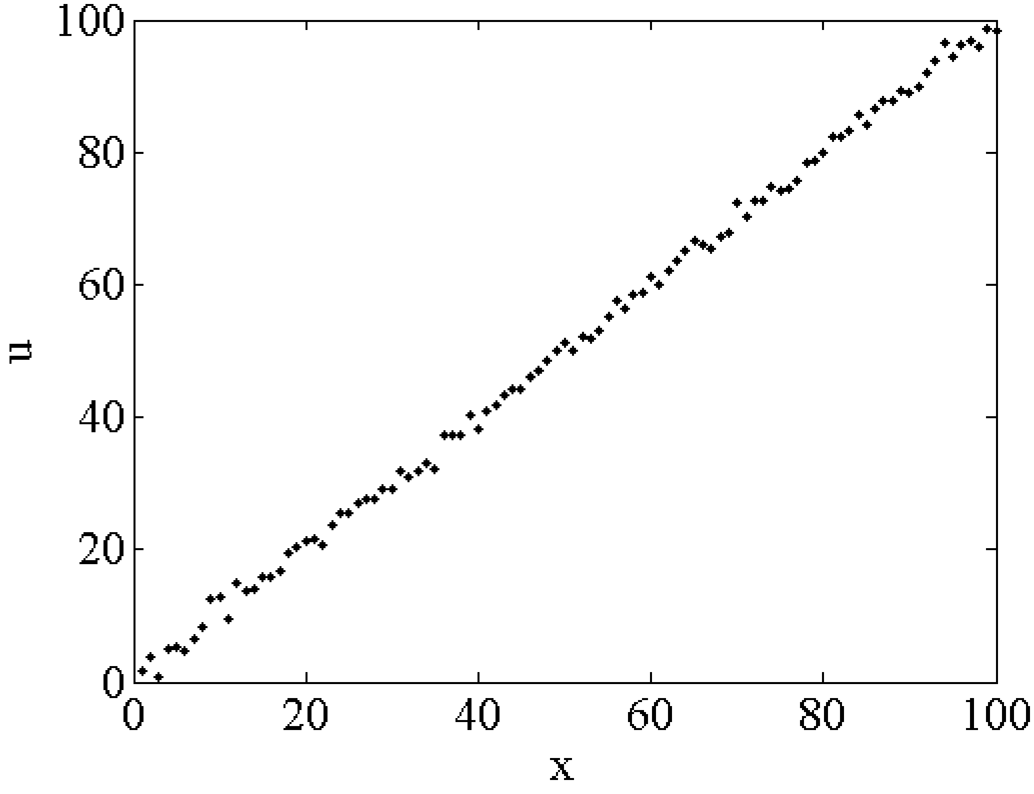
- Means that η is spatially uncorrelated

$$\left\langle \eta(x_i)\eta(x_j) \right\rangle = \delta(x_i - x_j)\sigma^2$$

- Its pair correlation function is local (δ) in real space, and hence uniform over all frequencies in Fourier space, similar in some way to “white” light

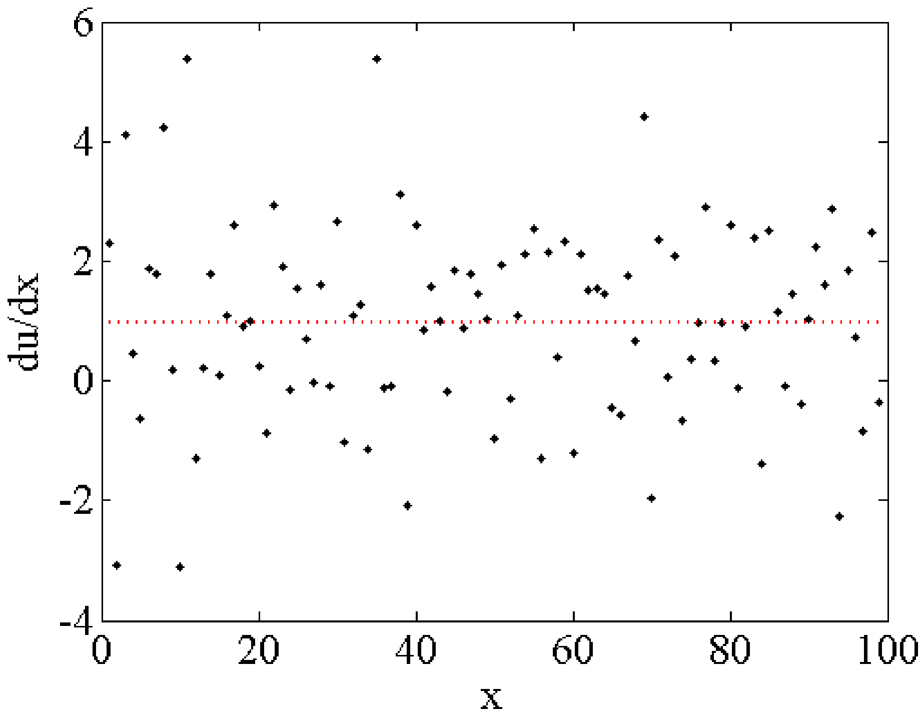
1D Problem

Find slope (strain) ε



Procedure 1

Macroscopic strain is the average of the local strain weighted by its length (/ area / volume)



$$\varepsilon \approx 0.97$$

Procedure 1

- Note that this estimate is solely based on the two extreme measurements

$$\langle \varepsilon \rangle = \frac{1}{N\Delta}\left((u_1 - u_0) + (u_2 - u_1) + (u_3 - u_2) + \cdots + (u_N - u_{N-1})\right)$$

Procedure 1

- Note that this estimate is solely based on the two extreme measurements

$$\langle \varepsilon \rangle = \frac{1}{N\Delta}\left((\cancel{u_1} - u_0) + (\cancel{u_2} - \cancel{u_1}) + (\cancel{u_3} - \cancel{u_2}) + \cdots + (u_N - \cancel{u_{N-1}})\right)$$

$$\langle \varepsilon \rangle = \frac{(u_N - u_0)}{N\Delta}$$

- Slope of line connecting first to last points!

Procedure 1

- Linear estimate in displacement data

$$\varepsilon_1 = \frac{(u_N - u_0)}{N\Delta} = \{\mathbf{e}_1\} \cdot \{\mathbf{u}\}$$

$$\{\mathbf{e}_1\}^{\mathsf{T}} = \frac{1}{N\Delta}(-1,0,0,\ldots,0,1)$$ Estimator

- Noise affecting this estimate obtained from linearity

$$\delta\varepsilon_1 = \frac{(\eta_N - \eta_0)}{N\Delta}$$

Gaussian distribution!

Procedure 1

- Unbiased procedure since

$$\langle \delta\varepsilon_1 \rangle = \frac{(\langle \eta \rangle - \langle \eta \rangle)}{N\Delta} = 0$$

- True for **all** linear estimators

$$\langle \delta\varepsilon_1 \rangle = \langle \{\mathbf{e}\}^\top \{\boldsymbol{\eta}\} \rangle = \{\mathbf{e}\}^\top \langle \{\boldsymbol{\eta}\} \rangle = 0$$

Procedure 1

- Variance of this estimate

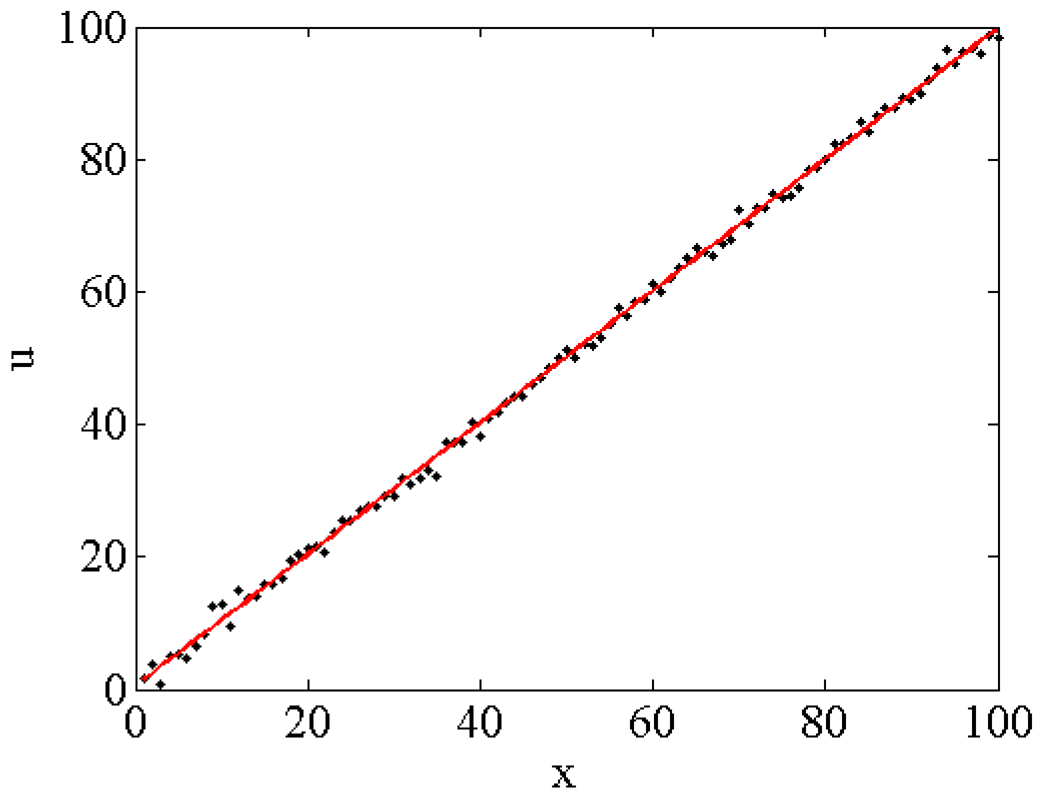
$$\langle \delta\varepsilon_1^2 \rangle = \frac{\left(\langle \eta_N^2 \rangle - 2\langle \eta_N \eta_0 \rangle + \langle \eta_0^2 \rangle\right)}{N^2 \Delta^2} = \frac{2\sigma^2}{N^2 \Delta^2}$$

- For all linear estimators

$$\langle \delta\varepsilon^2 \rangle = \langle e_i \eta_i \eta_j e_j \rangle = \sigma^2 \|\{\mathbf{e}\}\|^2$$

Procedure 2

- Linear regression



$$\varepsilon \approx 0.993$$

Procedure 2

Linear regression: minimization of quadratic difference between data and model

$$\mathcal{L}(a,b) = \sum_{i=0}^{N} (u_i - a x_i - b)^2$$

$$(\varepsilon, v) = Argmin_{a,b}\, \mathcal{L}(a,b)$$

$$\textstyle\sum_{i=0}^{N} (u_i - a x_i - b)\, x_i = 0$$

$$\textstyle\sum_{i=0}^{N} (u_i - a x_i - b) = 0$$

Procedure 2

- Solution is straightforward

$$\varepsilon = \frac{\langle ux\rangle - \langle u\rangle\langle x\rangle}{\langle x^2\rangle - \langle x\rangle^2}$$

$$v = \langle u\rangle - \varepsilon\langle x \rangle$$

- Linearity

$$\varepsilon_2 = \{\mathbf{e}_2\}^\mathsf{T}\{\mathbf{u}\} \qquad \{\mathbf{e}_2\} = \frac{1}{N}\frac{\{\mathbf{x}\} - \langle\{\mathbf{x}\}\rangle}{\langle\{\mathbf{x}^2\}\rangle - \langle\{\mathbf{x}\}\rangle^2}$$

- Thus

$$\langle\delta\varepsilon\rangle = \{\mathbf{e}_2\}^\mathsf{T}\langle\{\boldsymbol{\eta}\}\rangle = 0 \qquad \langle\delta\varepsilon^2\rangle = \left\langle e_i\eta_i\eta_j e_j\right\rangle = \sigma^2\|\{\mathbf{e}_2\}\|^2$$

Two (Excellent) Reasons for Choosing Procedure 2

- ✓ Bayesian justification
- ✓ Optimal extractor

Bayesian Justification

- According to model

$$\eta_i = u_i - ax_i - b$$

- Probability of obtaining this set of η

$$P \propto \prod_{i=1}^{N} p(\eta_i) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \exp\left(-\frac{\sum_i \eta_i^2}{2\sigma^2}\right)$$

- Maximum likelihood coincides with minimization of $\mathcal{L}(a, b)$

$$\mathcal{L}(a, b) = \sum_i \eta_i^2 = \sum_i (u_i - ax_i - b)^2$$

Bayesian Justification

- Generalization to noise
  - having heterogeneous variance
  - having arbitrary covariance (non-white)
  - not being Gaussian
  - etc.

can easily be handled this way

2nd Reason: Extractor

- What is expected from an extractor?

- Since the signal is a combination of two "trends," $\{\mathbf{1}\}$ and $\{\mathbf{x}\}$, one useless and one of interest, it is requested

$$\begin{aligned} \{\mathbf{e}\}^{\mathsf{T}}\{\mathbf{1}\} &= 0 \\ \{\mathbf{e}\}^{\mathsf{T}}\{\mathbf{x}\} &= 1 \end{aligned} \qquad \text{(A)}$$

- Any such $\{\mathbf{e}\}$ is an *unbiased* linear extractor

Optimal Extractor

- Uncertainty

$$\langle \delta\varepsilon^2 \rangle = \sigma^2 \|\{\mathbf{e}\}\|^2$$

- The **optimal** extractor is the one that minimizes the uncertainty

$$\{\mathbf{e}\}^* = \underset{\{\mathbf{e}\}^\top\{\mathbf{1}\}=0\,,\,\{\mathbf{e}\}^\top\{\mathbf{x}\}=1}{\operatorname{Argmin}} \|\{\mathbf{e}\}\|^2 \qquad \text{(B)}$$

- Using Lagrange multipliers, the functional to minimize is

$$\mathcal{B}(\{\mathbf{e}\}) = \|\{\mathbf{e}\}\|^2 - \lambda(\{\mathbf{e}\}^\top\{\mathbf{1}\}) - \mu(\{\mathbf{e}\}^\top\{\mathbf{x}\} - 1)$$

Optimal Extractor

- Minimization of (B) leads to

$$2e_i = \lambda + \mu x_i$$

and extractor constraints (A) add two conditions

$$\lambda = -\mu\langle\{\mathbf{x}\}\rangle$$

$$\lambda\langle\{\mathbf{x}\}\rangle + \mu\langle\{\mathbf{x}^2\}\rangle = 2/N$$

- So that

$$\mu = \frac{2}{N(\langle\{\mathbf{x}^2\}\rangle - \langle\{\mathbf{x}\}\rangle^2)}$$

$$\lambda = \frac{-2\langle\{\mathbf{x}\}\rangle}{N(\langle\{\mathbf{x}^2\}\rangle - \langle\{\mathbf{x}\}\rangle^2)}$$

$$\{\mathbf{e}\}^* = \frac{\{\mathbf{x}\} - \langle\{\mathbf{x}\}\rangle}{N(\langle\{\mathbf{x}^2\}\rangle - \langle\{\mathbf{x}\}\rangle^2)}$$

identical to $\{\mathbf{e}_2\}$

Extension

Uneven variance for Gaussian white noise

$$\mathcal{L}(a,b) = \sum_i \frac{\eta_i^2}{\sigma_i^2}$$

Extension

- Nonwhite Gaussian noise with covariance $[\mathbf{Cov}_{\boldsymbol{\eta}}]$

$$\mathcal{L}(a,b) = \sum_{i,j} \eta_i \mathrm{Cov}_{\mathrm{ij}}^{-1} \eta_j \equiv \|\{\boldsymbol{\eta}\}\|^2$$

Shown by diagonalization of the covariance matrix, and writing η in the eigen basis where noise is now white

- The inverse covariance defines the proper **metric** to evaluate residuals

→ Mahalanobis distance

Application

- Back to local strain average
- “Differentiation amplifies noise”: really?
- Not quite, but it creates correlations that should be disentangled to obtain the optimal answer

Application

- Local strain $\varepsilon_i = (u_i - u_{i-1})/\Delta$ is affected by noise $\lambda_i = (\eta_i - \eta_{i-1})/\Delta$

$$\begin{aligned}\mathrm{cov}_{ij} &= \langle \lambda_i \lambda_j \rangle \\ &= \frac{1}{\Delta^2} \langle (\eta_i - \eta_{i-1})(\eta_j - \eta_{j-1}) \rangle\end{aligned}$$

$$\langle \lambda_i \lambda_j \rangle = \frac{\sigma^2}{\Delta^2} \times \begin{cases} 2 & & |i-j| = 0 \\ -1 & \text{if} & |i-j| = 1 \\ 0 & & |i-j| > 1 \end{cases}$$

Application

When the optimal extractor is used for the functional

$$\mathcal{L}(a) = \|a\{\mathbf{1}\} - \{\boldsymbol{\varepsilon}\}\|^2$$

$$= \sum_{i,j}(a - \varepsilon_i)\mathrm{cov}_{ij}^{-1}(a - \varepsilon_j)$$

the optimal estimate of the strain is retrieved!

Exercise

- Show that any arbitrary (invertible) linear transform of the initial data $\{\mathbf{u}\}$ to $\{\mathbf{w}\} = [\mathbf{L}]\{\mathbf{u}\}$, does not affect the optimal estimate of the strain provided the appropriate metric $[\mathbf{cov}^{-1}]$ is used

CONSEQUENCES FOR DIC

DIC Notations

- Images $f(\underline{\mathbf{x}})$ (reference) and $g(\underline{\mathbf{x}})$ (deformed)
- Displacement field $\underline{\mathbf{v}}(\underline{\mathbf{x}})$
- Define corrected images as $\tilde{g}_{\mathbf{v}}(\underline{\mathbf{x}}) = g(\underline{\mathbf{x}} + \underline{\mathbf{v}}(\underline{\mathbf{x}}))$ and residuals $\rho_{\mathbf{v}}(\underline{\mathbf{x}}) = \tilde{g}_{\mathbf{v}}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}})$
- Standard DIC reads

$$\underline{\mathbf{u}} = \mathrm{Argmin}_{\mathbf{v}} \, \|\rho_{\mathbf{v}}\|^2$$

Consequences for DIC

Considering that images are affected by Gaussian white noise, the **optimal** cost function is the quadratic difference between images

$$\|\rho\|^2 = \frac{1}{\sigma^2} \sum_{\underline{\mathbf{x}}} \rho(\underline{\mathbf{x}})^2$$

Poisson Noise

- More than often noise variance varies with gray level
- For Poisson noise $\sigma^2 \propto f$
- *Optimal* DIC functional is

Absolute gray level!

$$\|\rho\|^2 \propto \sum_{\underline{\mathbf{x}}} \frac{\left(\tilde{g}(\underline{\mathbf{x}}) - f(\underline{\mathbf{x}})\right)^2}{f(\underline{\mathbf{x}})}$$

Poisson Noise

- Anscombe 'trick' $f(\underline{\mathbf{x}}) \leftarrow \sqrt{f(\underline{\mathbf{x}})}$
- Poisson noise becomes Gaussian with uniform variance
- *Optimal* DIC functional

$$\|\rho\|^2 \propto \sum_{x} \left(\sqrt{\tilde{g}(\underline{\mathbf{x}})} - \sqrt{f(\underline{\mathbf{x}})} \right)^2$$

$$\propto \sum_{x} \left(\widetilde{\sqrt{g(\underline{\mathbf{x}})}} - \sqrt{f(\underline{\mathbf{x}})} \right)^2$$

[Sur *et al*., 2015, *SIAM J. Imaging Sci*. 8(1) pp. 611-643]

[$\mathbf{Cov_u}$] for DIC

- **Local DIC**: Covariance is trivial when subsets do not overlap (white noise) provided no smoothing interpolation is used to deliver a nice-looking result

- **Global DIC**: Inverse covariance is a costless ingredient (Hessian of the cost function)

[HF *et al*., 2012, *Exp. Mech*. 52(9) pp.1503-1519]

Time Series in DIC

- Time series
- Images $f(\underline{\mathbf{x}}, t)$
- Displacement field from t_0 $\underline{\mathbf{u}}(\underline{\mathbf{x}}, t)$
- Define corrected images as

$$\tilde{f}(\underline{\mathbf{x}}, t) = f(\underline{\mathbf{x}} + \underline{\mathbf{u}}(\underline{\mathbf{x}}, t), t)$$

and residuals

$$\rho(\underline{\mathbf{x}}, t) = \tilde{f}(\underline{\mathbf{x}}, t) - f(\underline{\mathbf{x}}, t_0)$$

Time Series in DIC

- Standard DIC

$$\underline{\mathbf{u}}(\underline{\mathbf{x}}, t) = \mathrm{Argmin} \sum_{\underline{\mathbf{x}}, t} \rho(\underline{\mathbf{x}}, t)^2$$

- However, noise affects the reference image at t_0 for all t in ρ
- Noise on residuals is temporally correlated

Time Series in DIC

- Covariance of noise on ρ:

$$\lambda(\underline{\mathbf{x}}, t_i) = \eta(\underline{\mathbf{x}}, t_i) - \eta(\underline{\mathbf{x}}, t_0)$$

$$\begin{aligned} \mathrm{cov}_{ij} &= \langle \lambda(\underline{\mathbf{x}}, t_i)\lambda(\underline{\mathbf{x}}, t_j)\rangle \\ &= \sigma^2(1 + \delta_{ij}) \end{aligned}$$

$$[\mathbf{Cov}_{\boldsymbol{\lambda}}] = \sigma^2([\mathbf{1}] + [\boldsymbol{I}])$$

- Inverse covariance

$$[\mathbf{Cov}_{\boldsymbol{\lambda}}]^{-\mathbf{1}} = \frac{1}{\sigma^2}\left([\boldsymbol{I}] - \frac{1}{N+1}[\mathbf{1}]\right)$$

Time Series in DIC

- Thus optimal DIC for time series should be based on the minimization of $||\rho||^2$ using the temporal $[\mathbf{Cov_\lambda}]^{-1}$ metric

- After some simple algebra

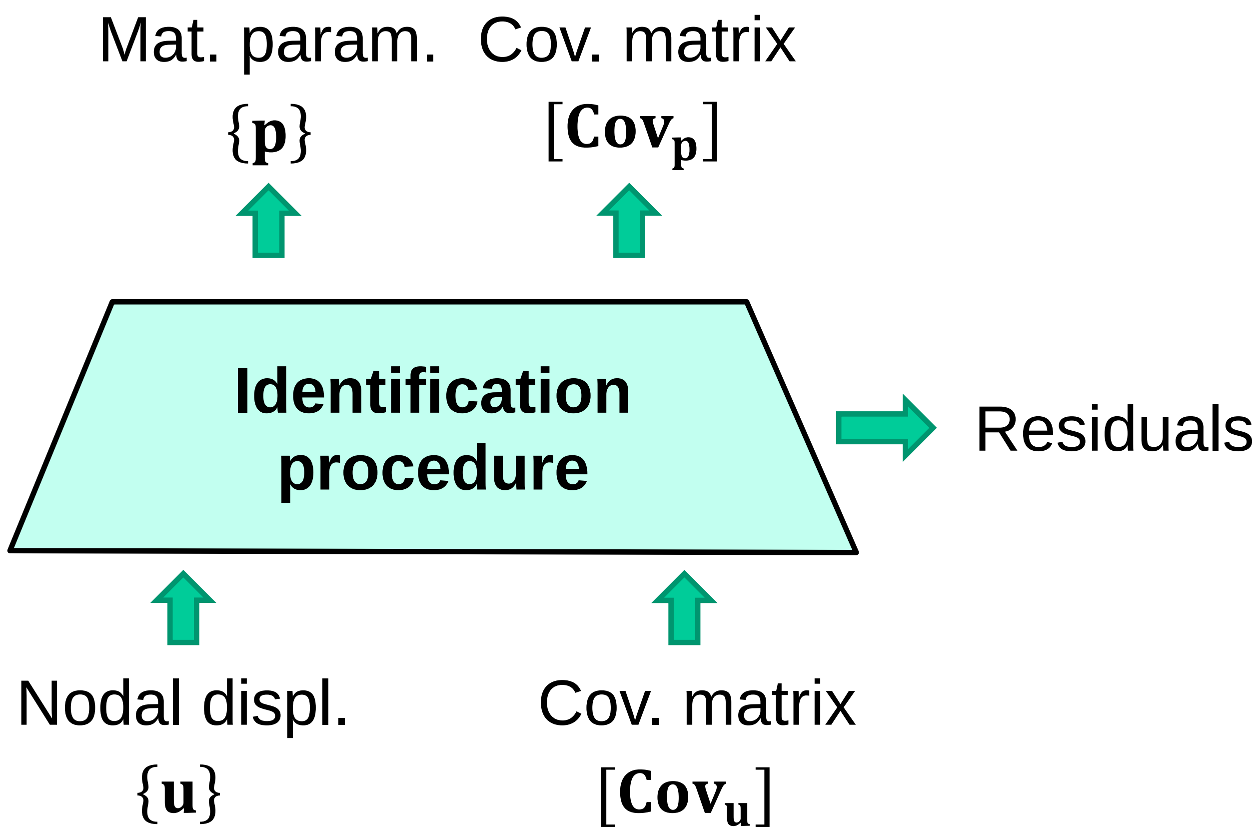
$$||\rho||^2 = \sum_{\underline{\mathbf{x}},t} \left(\tilde{f}\left(\underline{\mathbf{x}}, t\right) - \hat{f}_{\mathrm{ref}}\left(\underline{\mathbf{x}}\right) \right)^2$$

$$\hat{f}_{\mathrm{ref}}\left(\underline{\mathbf{x}}\right) = \frac{1}{(N+1)} \sum_{t} \tilde{f}\left(\underline{\mathbf{x}}, t\right)$$

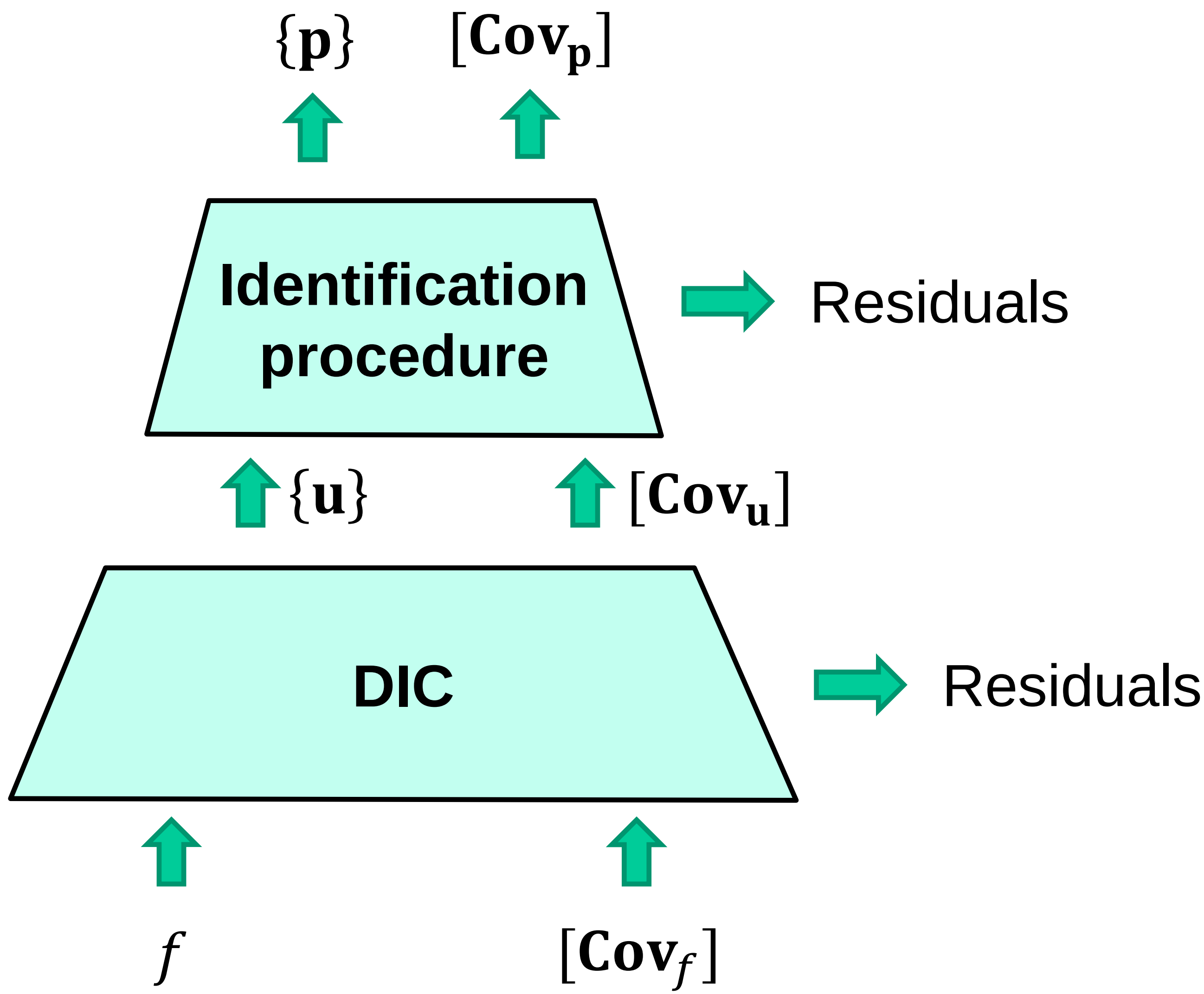
[Berny *et al.*, 2018, *Opt. Lasers Eng*. 111 pp. 135-153]

IDENTIFICATION

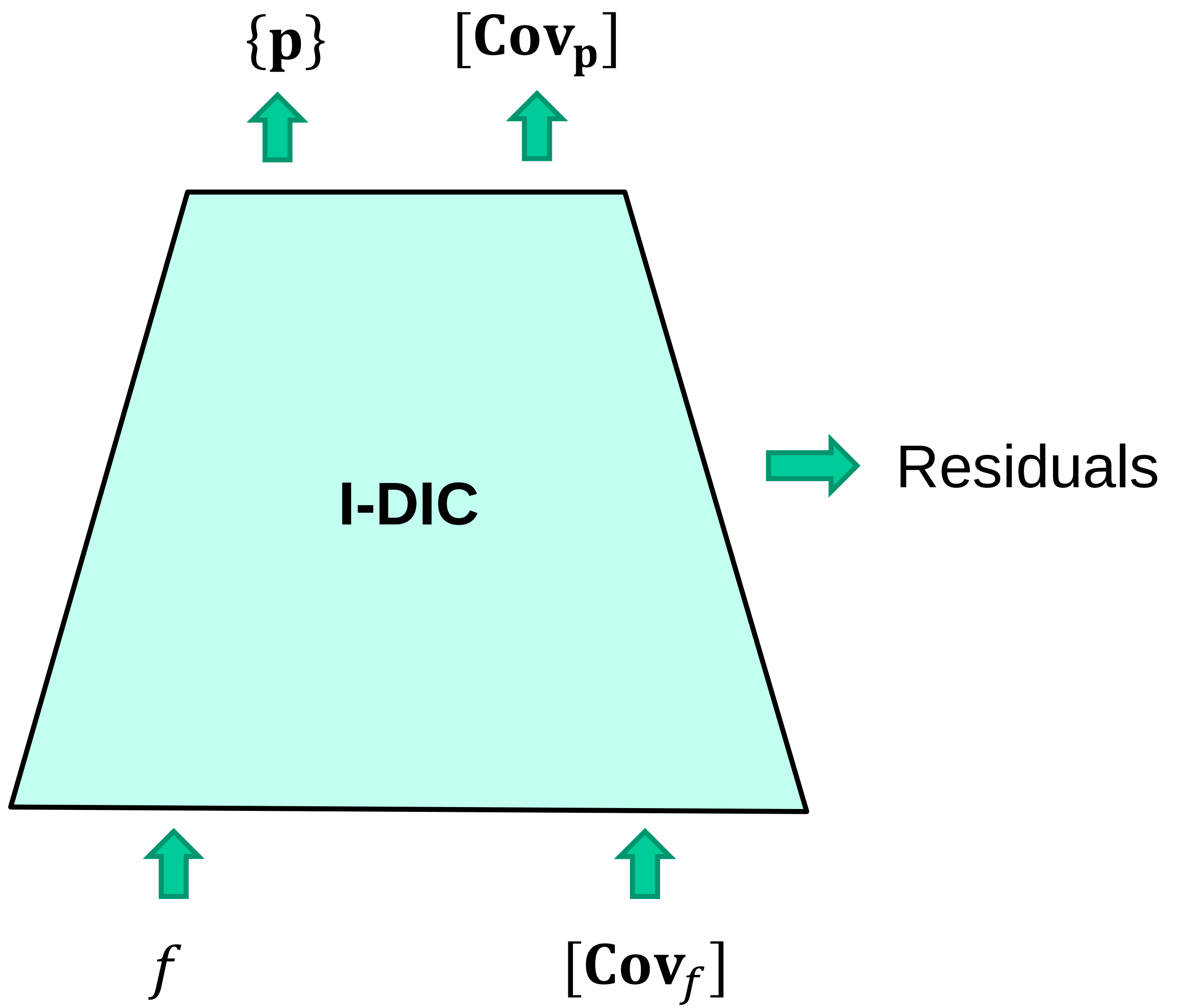
Identification



Chained Identification



Integrated DIC



Identification in Nutshell

- For linear elastic solids (to start off with) and FEA modeling
- DIC provides measured displacement field at nodes $\underline{\mathbf{u}}_i = \underline{\mathbf{u}}\left(\underline{\mathbf{x}}_i\right)$
- DIC provides a covariance matrix for the measured displacement Cov_{ij}
- Nodes belong to bulk (in), Neumann (Nbc), or Dirichlet boundary (Dbc)
- Nodal forces (usually and here $\{\boldsymbol{f}\} = \{\mathbf{0}\}$) are known on $(in) \cup (Nbc)$

Identification in a Nutshell

- One looks for parameters $\{\mathbf{p}\}$ from which one can compute a rigidity matrix $[\boldsymbol{K}(\{\mathbf{p}\})]$ such that the computed displacement field $\{\mathbf{v}\}$ obeys

$$K_{ij}v_j = 0 \qquad i \in (in) \cup (Nbc)$$
$$v_i = u_i \qquad i \in (Dbc)$$

- $\{\mathbf{p}\}$ is adjusted so that $\{\mathbf{v}\}$ matches $\{\mathbf{u}\}$ at best

[Avril *et al*., 2008, *Exp. Mech*. 48 (4) pp. 381-402]

Notations

- Denote $[\boldsymbol{D}]$ diagonal matrix such that

$$D_{ii} = 1 \qquad i \in (in) \cup (Nbc)$$
$$D_{ii} = 0 \qquad i \in (Dbc)$$

- Solution to linear elastic problem

$$K_{ij} v_j = 0 \qquad i \in (in) \cup (Nbc)$$
$$v_i = u_i \qquad i \in (Dbc)$$

linear in its Dirichlet b.c. denoted

$$\{\mathbf{v}\} \equiv [\boldsymbol{L}]([\boldsymbol{I}] - [\boldsymbol{D}])\{\mathbf{u}\}$$

- Properties

$$[\boldsymbol{D}][\boldsymbol{K}]\{\mathbf{v}\} = \{\mathbf{0}\} \qquad ([\boldsymbol{I}] - [\boldsymbol{D}])(\{\mathbf{u}\} - \{\mathbf{v}\}) = \{\mathbf{0}\}$$

$$[\boldsymbol{D}][\boldsymbol{K}][\boldsymbol{L}]([\boldsymbol{I}] - [\boldsymbol{D}])\{\mathbf{u}\} = \{\mathbf{0}\}$$

FEMU

- Finite Element Model Updating consists in minimizing the L2 norm of the difference between measured $\{\mathbf{u}\}$ and computed $\{\mathbf{v}\}$ displacements with respect to $\{\mathbf{p}\}$

$$\begin{aligned}\mathcal{F}(\{\mathbf{p}\}) &= \|\{\mathbf{u}\} - \{\mathbf{v}\}\|^2 \\ &= \|[\boldsymbol{I}] - [\boldsymbol{L}]([\boldsymbol{I}] - [\boldsymbol{D}])]\{\mathbf{u}\}\|^2\end{aligned}$$

[Friswell *et al*., 1995, *Finite Element Model Updating in Structural Dynamics*, Springer]

Equilibrium gap (EG)

- For measured displacement $\{\mathbf{u}\}$, nodal forces read

$$K_{ij}u_j = f_i$$

$$[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{f}\}$$

- Ideally, $f_i = 0 \quad \text{for} \quad i \in (in) \cup (Nbc)$

$$[\mathbf{D}]\{\mathbf{f}\} = [\mathbf{D}][\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{0}\}$$

- EG consists in minimizing the L2 norm of the unbalanced forces

$$\mathcal{F}(\{\mathbf{p}\}) = \|[\mathbf{D}][\mathbf{K}]\{\mathbf{u}\}\|^2$$
$$= \|[\mathbf{D}][\mathbf{K}](\{\mathbf{u}\} - \{\mathbf{v}\})\|^2$$

[Claire *et al*., 2004, *Int. J. Num. Meth. Eng*. 62 pp.189-208]

Constitutive law error (CLE)

- CLE consists in minimizing the *elastic energy* associated with unbalanced forces $[\mathbf{D}]\{\mathbf{f}\}$
- Using $[\mathbf{D}][\mathbf{K}]\{\mathbf{v}\} = \{\mathbf{0}\}$ and $([\mathbf{I}] - [\mathbf{D}])(\{\mathbf{u}\} - \{\mathbf{v}\}) = \{\mathbf{0}\}$ the *elastic energy* reads

$$\begin{aligned}\mathcal{F}(\{\mathbf{p}\}) &= (\{\mathbf{u}\} - \{\mathbf{v}\})^{\top}[\mathbf{D}][\mathbf{K}](\{\mathbf{u}\} - \{\mathbf{v}\}) \\ &= (\{\mathbf{u}\} - \{\mathbf{v}\})^{\top}[\mathbf{K}](\{\mathbf{u}\} - \{\mathbf{v}\})\end{aligned}$$

[Calloch *et al*., 2002, *Technol. Form*. 100 pp.36-41]
[Geymonat *et al*., 2002, *C.R. Mécanique* 330(6) pp. 403-408]

Virtual Fields Method (VFM)

- Because unbalanced forces should vanish, their virtual work with a set of chosen virtual displacement fields $\boldsymbol{\Psi}^n$ should be null

$$\{\boldsymbol{\Psi}^n\}^\top[\mathbf{D}][\mathbf{K}]\{\mathbf{u}\} = 0$$

$$\{\boldsymbol{\Psi}^n\}^\top[\mathbf{D}][\mathbf{K}](\{\mathbf{u}\} - \{\mathbf{v}\}) = 0$$

- Denoting $[\mathbf{H}] = \sum_n \{\boldsymbol{\Psi}^n\}^\top\{\boldsymbol{\Psi}^n\}$

  VF can be rewritten as minimization of

$$\mathcal{F}(\{\mathbf{p}\}) = \| [\mathbf{H}][\mathbf{D}][\mathbf{K}](\{\mathbf{u}\} - \{\mathbf{v}\}) \|^{\mathbf{2}}$$

[Grédiac, 1989, *C.R. Acad. Sci*. Série II 309(1) pp. 1-5]

Summary

FEMU $\mathcal{F}(\{\mathbf{p}\}) = (\{\mathbf{u}\} - \{\mathbf{v}\})^\top [\mathbf{I}](\{\mathbf{u}\} - \{\mathbf{v}\})$

EG $\mathcal{F}(\{\mathbf{p}\}) = (\{\mathbf{u}\} - \{\mathbf{v}\})^\top [\mathbf{K}][\mathbf{D}][\mathbf{K}](\{\mathbf{u}\} - \{\mathbf{v}\})$

CLE $\mathcal{F}(\{\mathbf{p}\}) = (\{\mathbf{u}\} - \{\mathbf{v}\})^\top [\mathbf{K}](\{\mathbf{u}\} - \{\mathbf{v}\})$

VF $\mathcal{F}(\{\mathbf{p}\}) = (\{\mathbf{u}\} - \{\mathbf{v}\})^\top [\mathbf{K}][\mathbf{D}][\mathbf{\Psi}]^\top [\mathbf{\Psi}][\mathbf{D}][\mathbf{K}](\{\mathbf{u}\} - \{\mathbf{v}\})$

REG $\mathcal{F}(\{\mathbf{p}\}) = (\{\mathbf{u}\} - \{\mathbf{v}\})^\top [\mathbf{K}][\mathbf{D}][\mathbf{N}]^\top [\mathbf{N}][\mathbf{D}][\mathbf{K}](\{\mathbf{u}\} - \{\mathbf{v}\})$

All! $\mathcal{F}(\{\mathbf{p}\}) = (\{\mathbf{u}\} - \{\mathbf{v}\})^\top [\mathbf{A}](\{\mathbf{u}\} - \{\mathbf{v}\})$

Question

- All the previous methods aimed at minimizing a norm (or semi-norm) of $(\{\mathbf{u}\} - \{\mathbf{v}\})$
- Which one is the best?
- The basic flaw of all these methods is that none accounts for the quality of the measured data

Optimal Identification

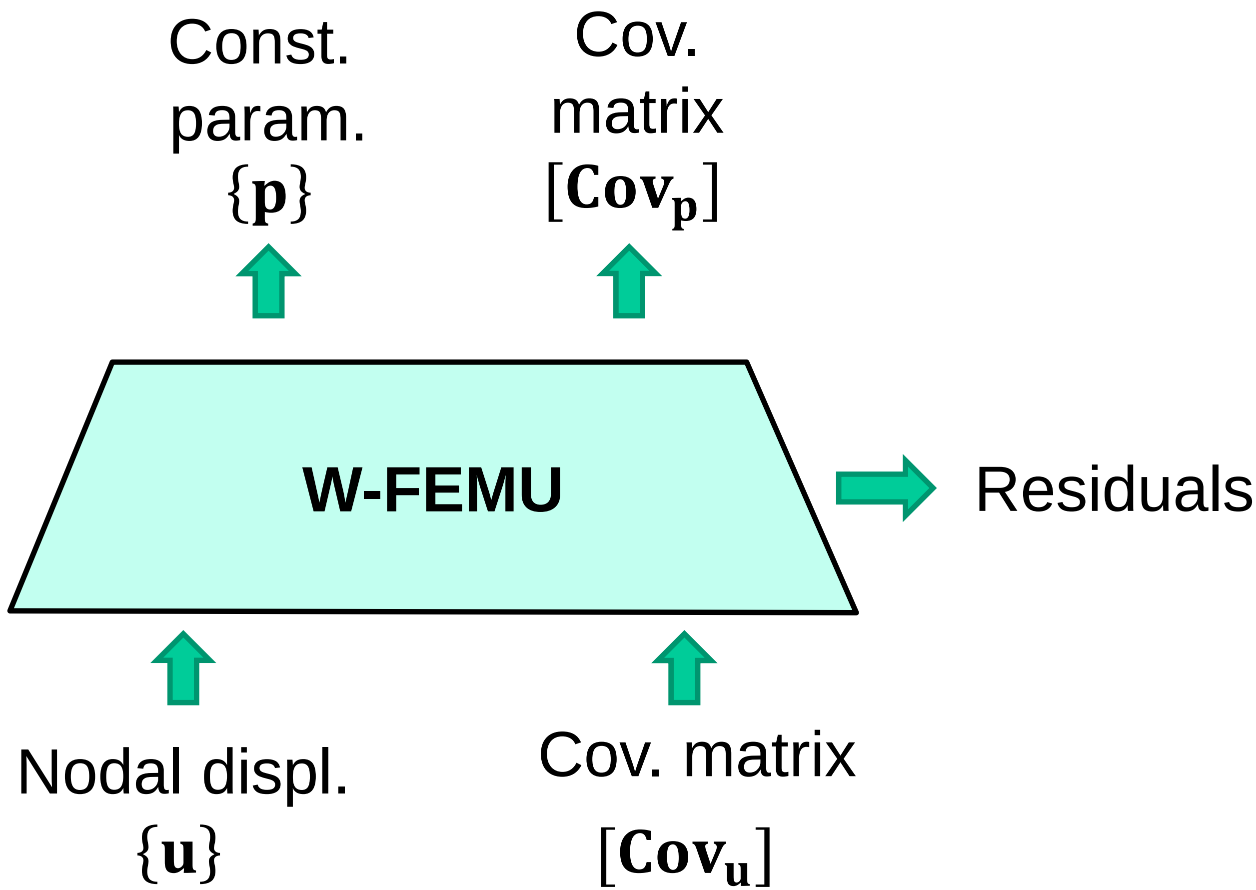
- From the covariance of the measured displacement, the *optimal* metric to evaluate the discrepancy between measured and computed displacement fields is

$$\mathcal{F}(\{\mathbf{p}\}) = (\{\mathbf{u}\} - \{\mathbf{v}\})^\top [\mathbf{Cov_u}]^{-\mathbf{1}} (\{\mathbf{u}\} - \{\mathbf{v}\})$$

- Bonus: uncertainty on $\{\mathbf{p}\}$
  (full covariance matrix)

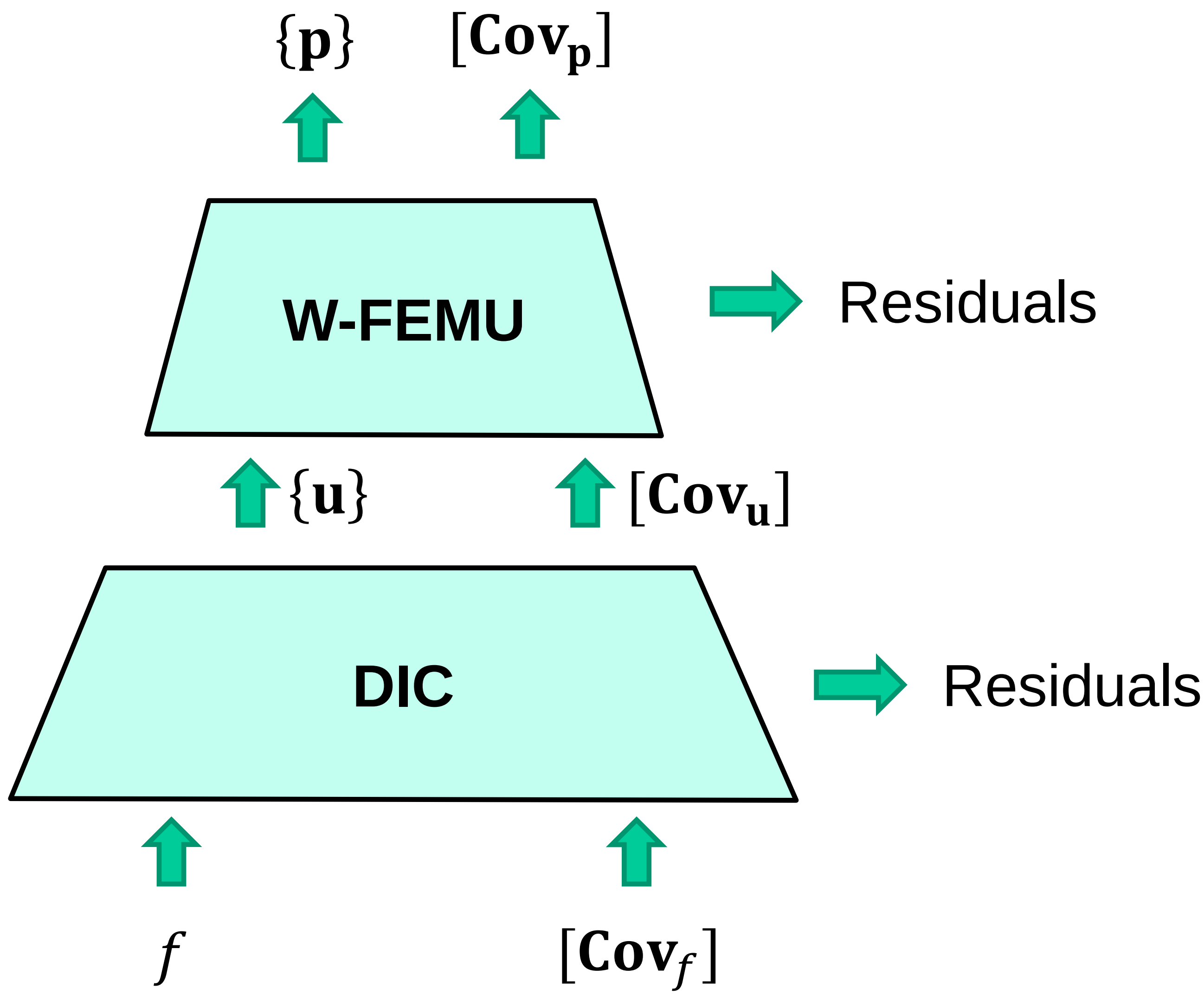
[SR *et al.*, 2019, *Int. J. Solids Struct. DOI: 10.1016/j.ijsolstr.2018.11.008*]

Weighted FEMU

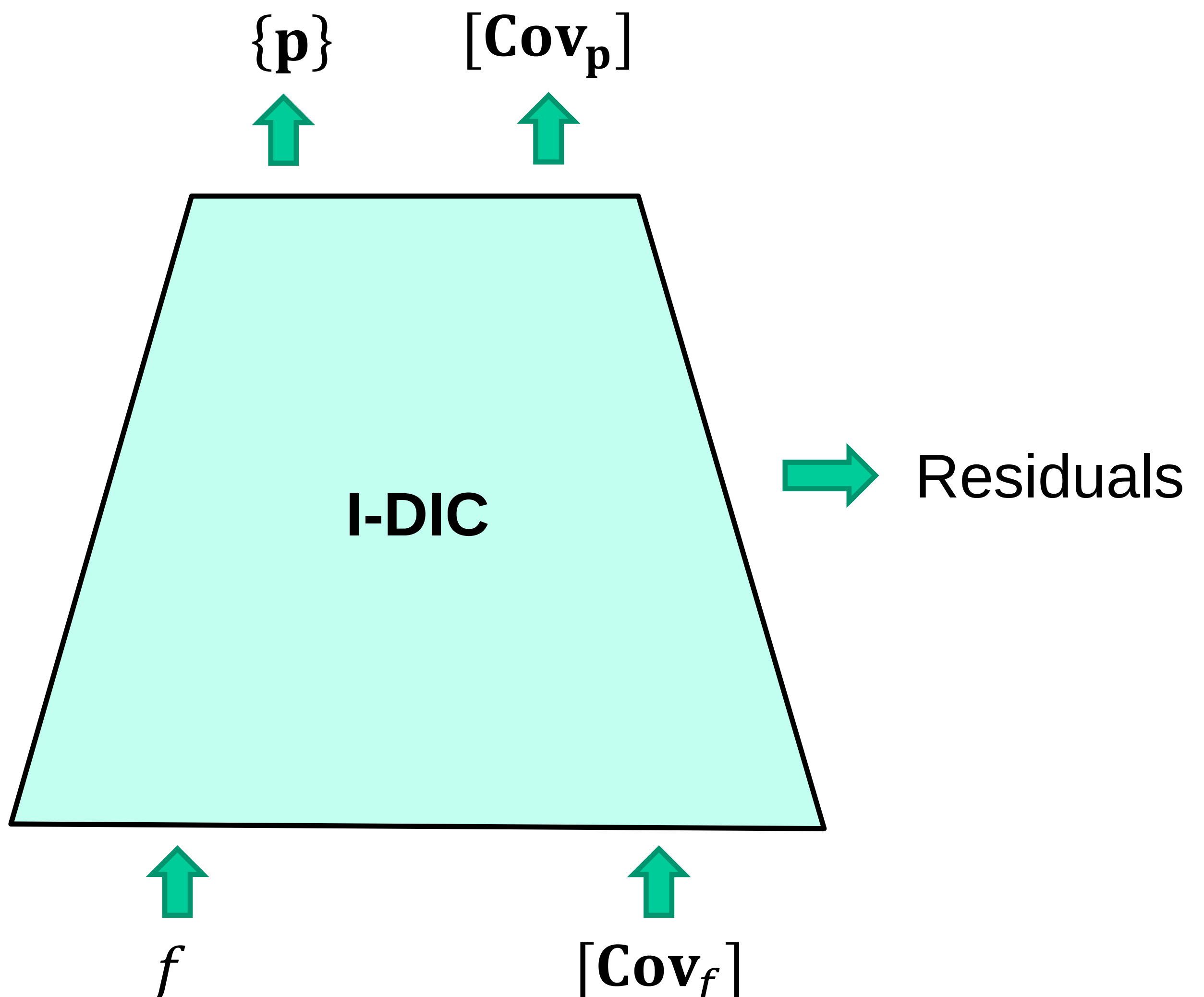


[Gras *et al.*, 2015, *Int. J. Solids Struct.* 55 pp. 2-16]

Chained Identification



Integrated DIC



[Leclerc *et al.*, 2009, *in LNCS* 5496 pp. 161-171]
[Mathieu *et al.*, 2015, *Exp. Mech.* 55(1) pp. 105-119]

Data Fusion in Optimal Identification

When some additional data are available, such as force measurements, they can be included as the quadratic difference between measured and computed data weighted by the inverse variance of the measurement

$$\mathcal{F}(\{\mathbf{p}\}) = (\{\mathbf{u}\} - \{\mathbf{v}\})^{\top}[\mathbf{Cov_u}]^{-1}\,(\{\mathbf{u}\} - \{\mathbf{v}\}) + \frac{1}{\sigma_{\mathrm{f}}^2}\left\|\{\mathbf{f_{meas}}\} - \{\mathbf{f_{comp}}\}\right\|^{\mathbf{2}}$$

- Likelihood provides a universal gauge

[Bertin *et al.*, 2016, *J. Strain Analysis* 51(2) pp. 118-131]

Data Fusion in Optimal Identification

- When considering a test with multiple images, the fusion rule is similar
- When combining multiple experiments together, the fusion rule is similar
- ***Note***: the inverse covariance weighting is transparent to the way data are partitioned / grouped / assembled

[Neggers *et al*., 2019, *Mech. Mat.* 133 pp. 71-84]

Bonus

- After identification, residuals $(\{\mathbf{u}\} - \{\mathbf{v}\})$ are extremely informative to assess adequacy of model to observed data
- They should reduce to noise within the above framework (driven by noise minimization)
- Often, residuals highlight weaknesses of modeling pointing toward ways to enhance the considered model

EXEMPLES

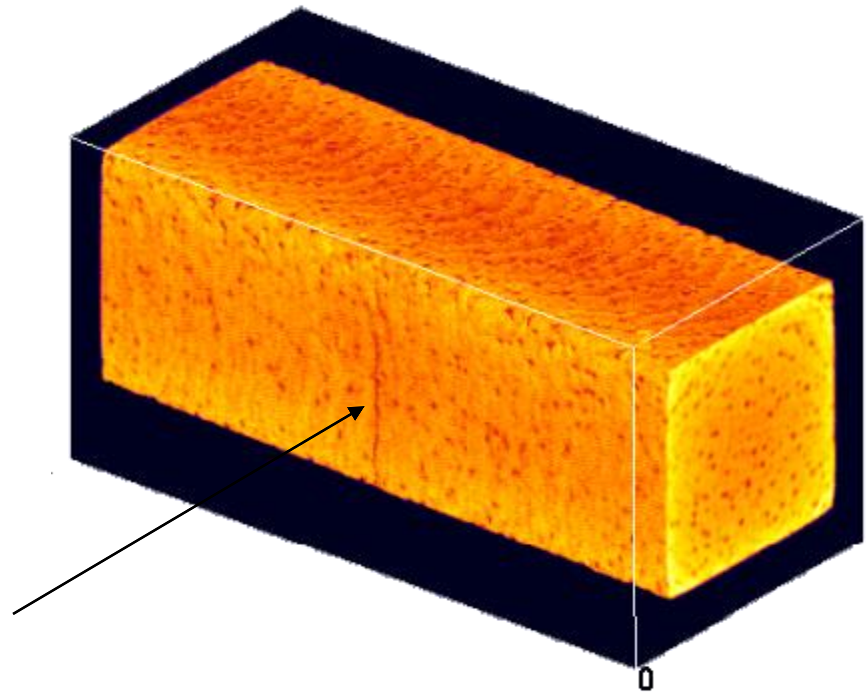
C8-DVC FOR VALIDATION

Images from X-Ray Tomography

- Nodular graphite cast iron
- Gauge volume: 2 × 4 × 2 mm^3
- Voxel size: 13.5 µm
- Loading:

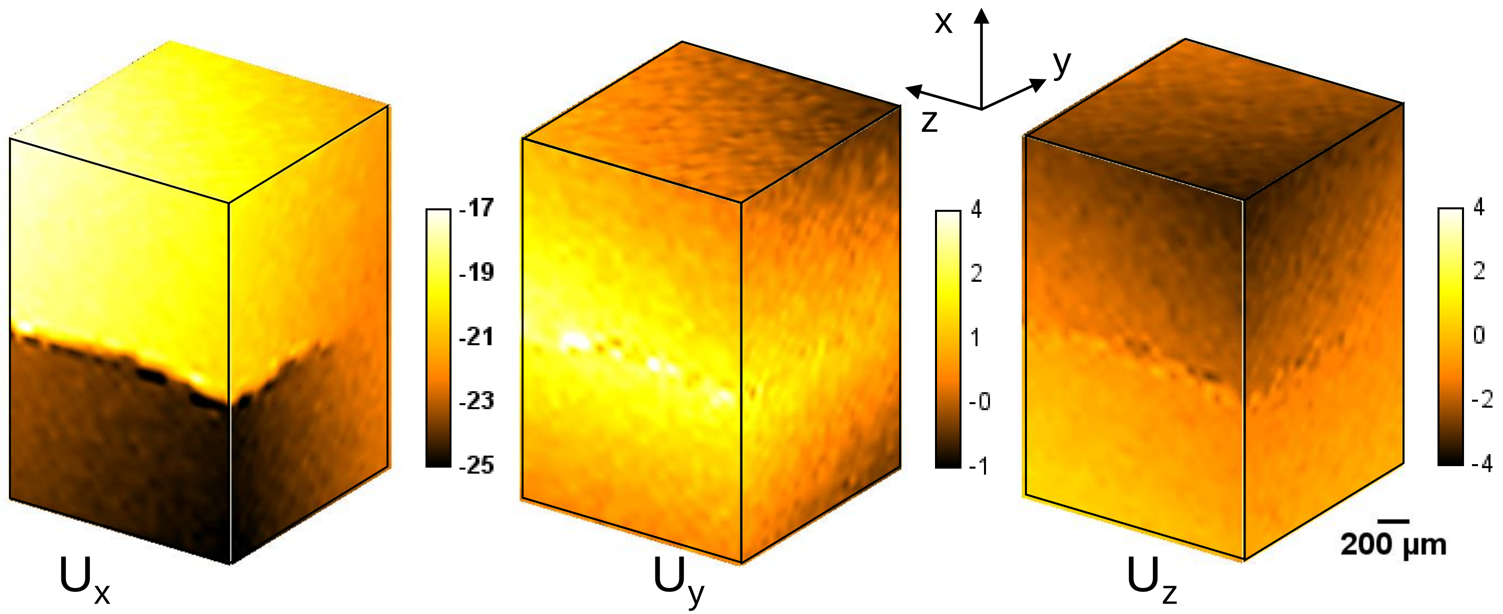
20	N	Reference
100	N	
200	N	
300	N	
400	N	Maximum load
20	N	Unload

- Fatigue precrack



[Réthoré *et al*., 2008, *C.R. Mécanique* 336 pp. 643-649]

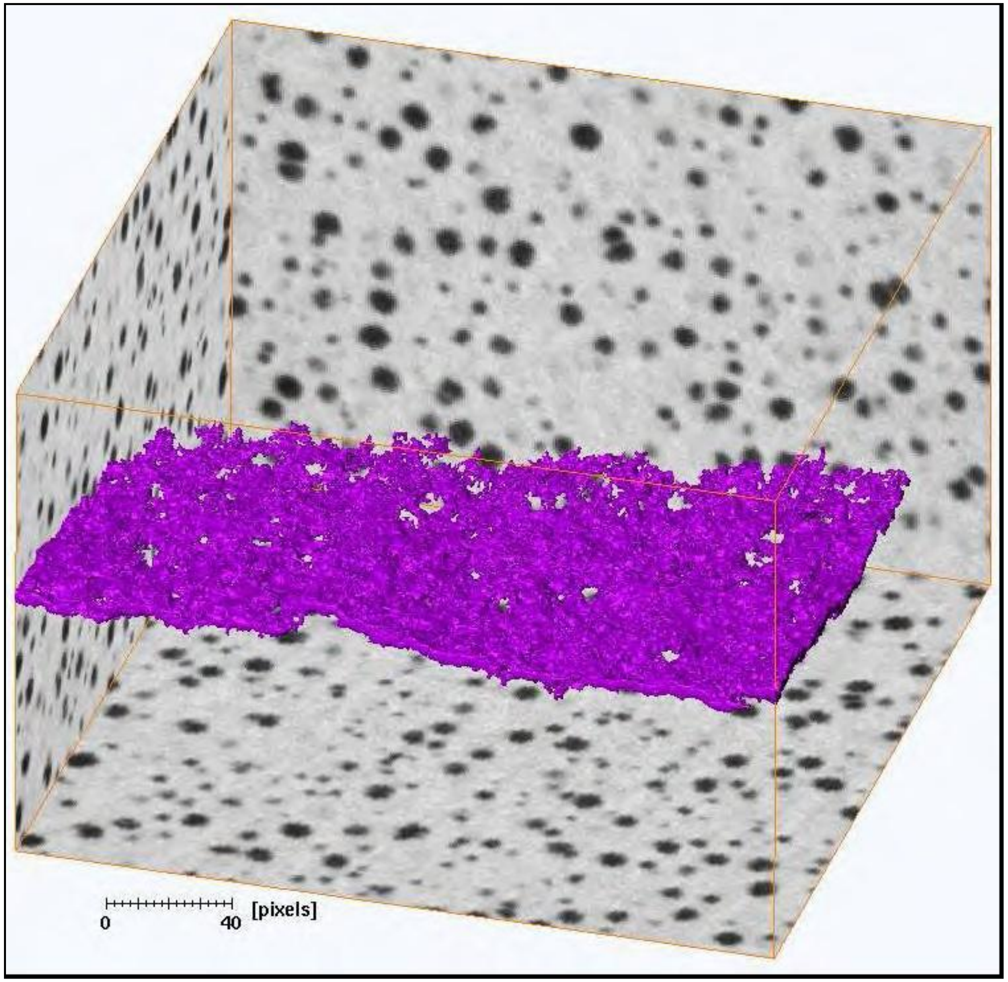
3D Displacement Fields*



1 voxel ↔ 3.5 µm

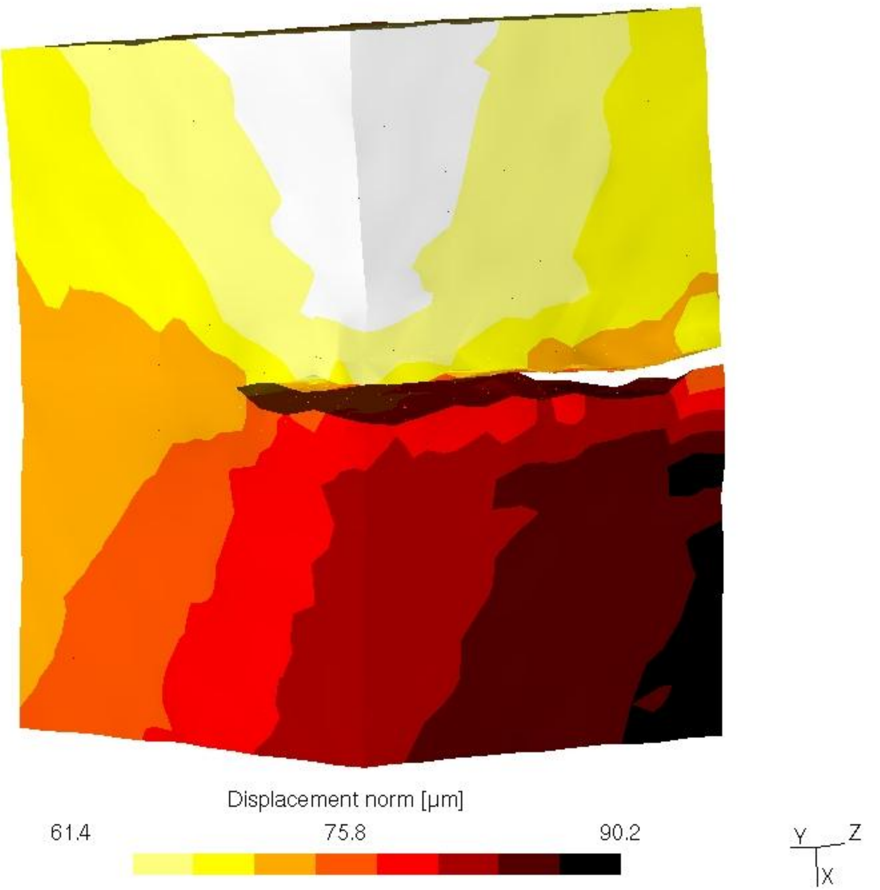
*[Limodin *et al.*, 2009, *Acta Mat.* 57 4090-4101]

Crack Shape Identification

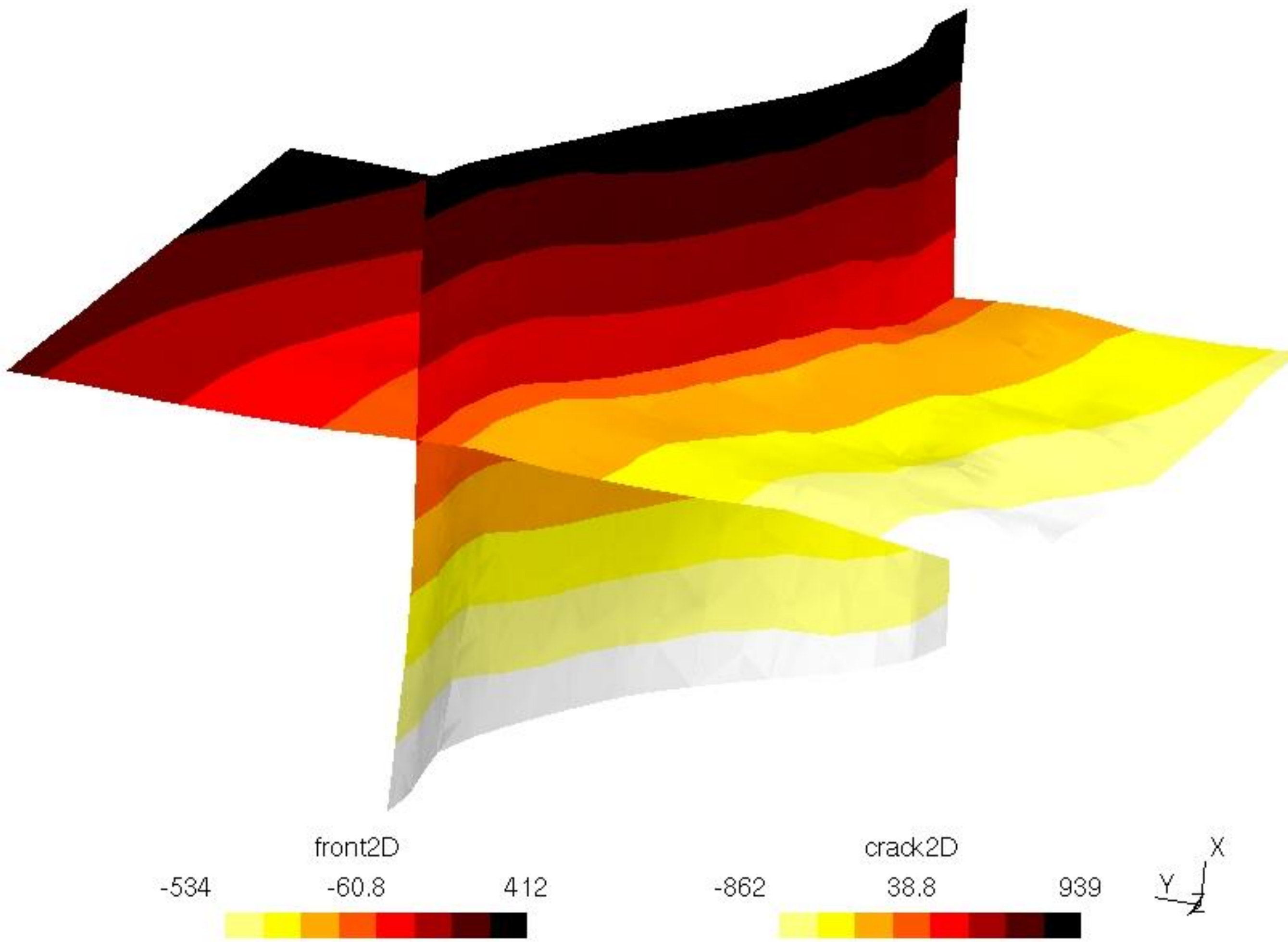


Correlation residual field $\eta(\underline{\mathbf{x}}) = \left| f(\underline{\mathbf{x}}) - g(\underline{\mathbf{x}} + \underline{\mathbf{u}}(\underline{\mathbf{x}}))\right|$

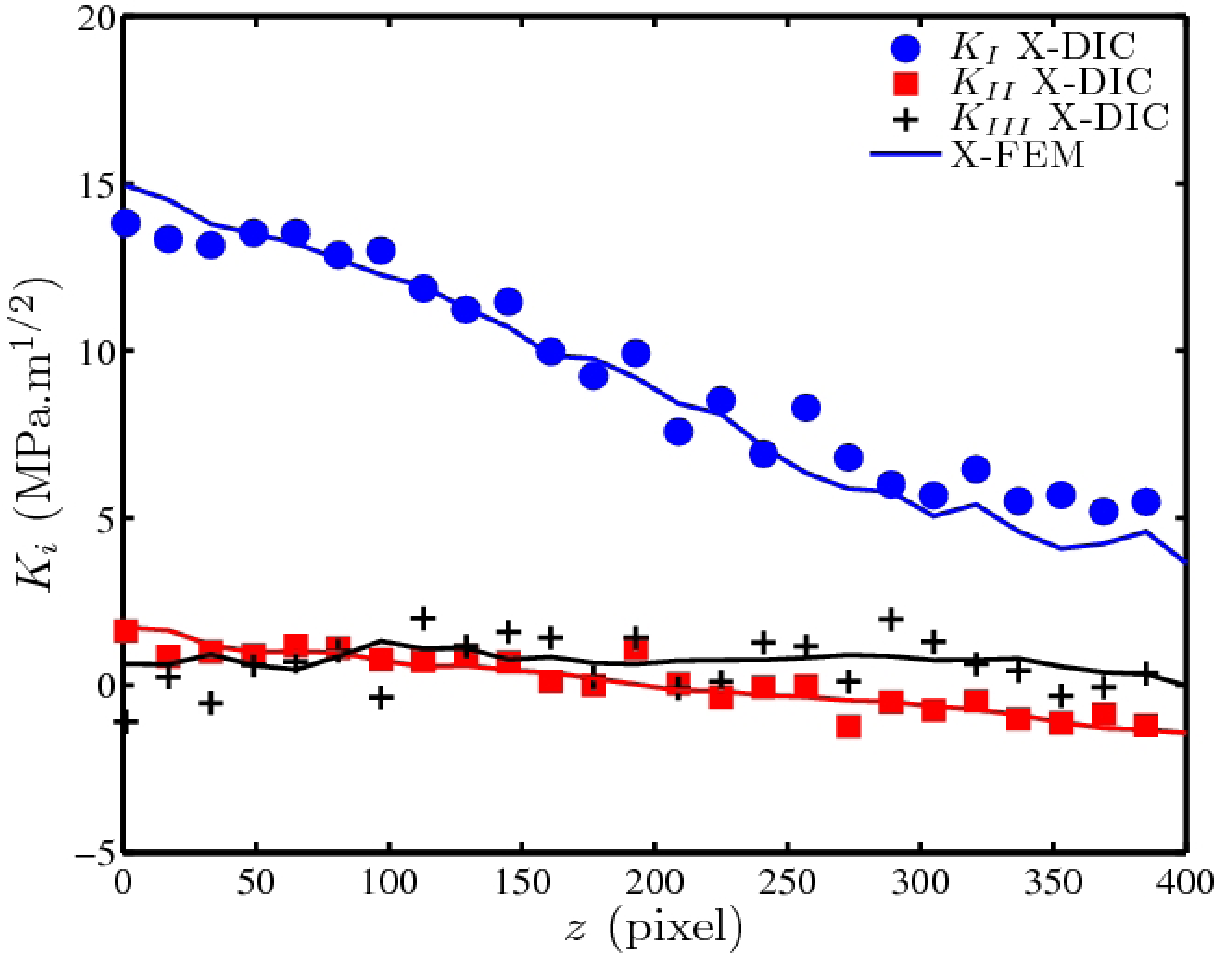
Deformed Shape (× 10)



Crack Front Identification



SIF Measurement



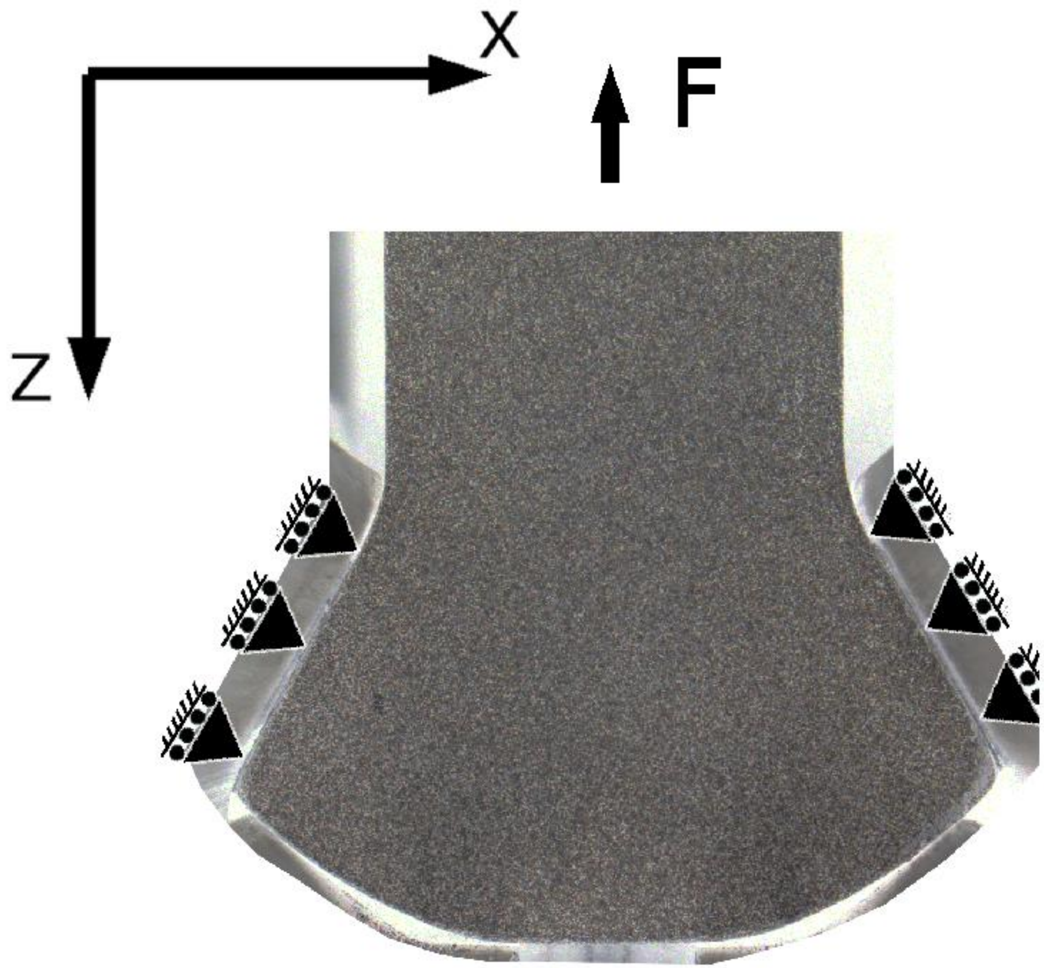
K_I
K_{II}
K_{III}

[Limodin *et al.*, 2009, *Acta Mat.* 57 pp. 4090-4101]
[Limodin *et al.*, 2010, *Acta Mat.* 58 pp. 2957-2967]
[Rannou *et al.*, 2010, *Comp. Meth. Appl. Mech. Eng.* 199 pp. 1307-1325]

DIC FOR IDENTIFICATION

Example: Fan Blade Root

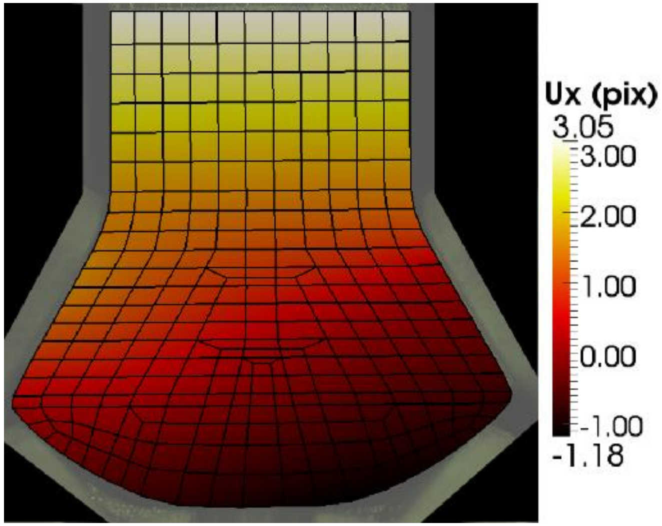
Interlock woven composite

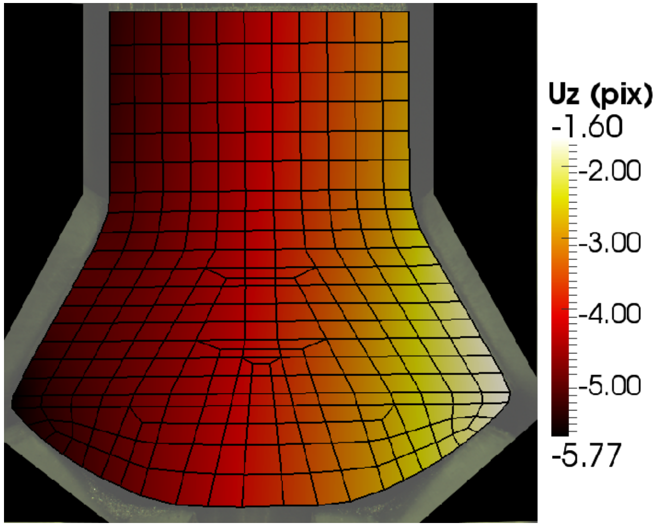


SAFRAN AIRCRAFT ENGINES

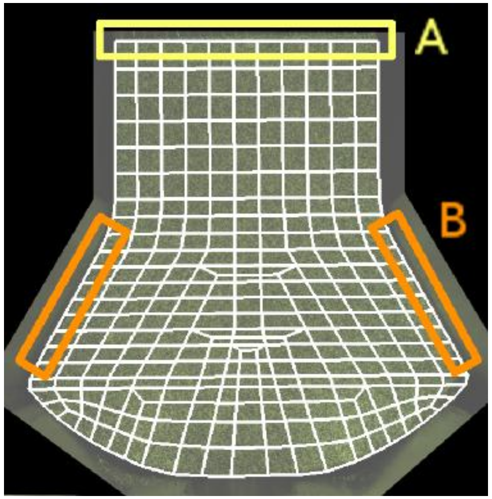
[Gras *et al.*, 2015, *Int. J. Solids Struct.* 55 pp. 2-16]

DIC Analysis



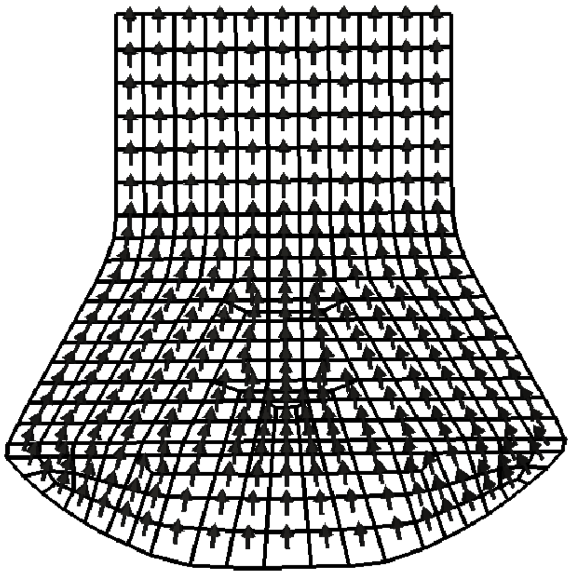


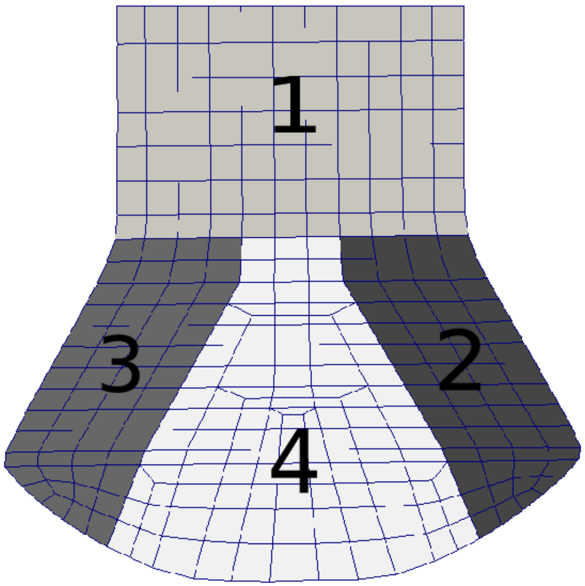
Note the unexpected breakdown of symmetry



Microstructure Model

Partition into four anisotropic elastic zones





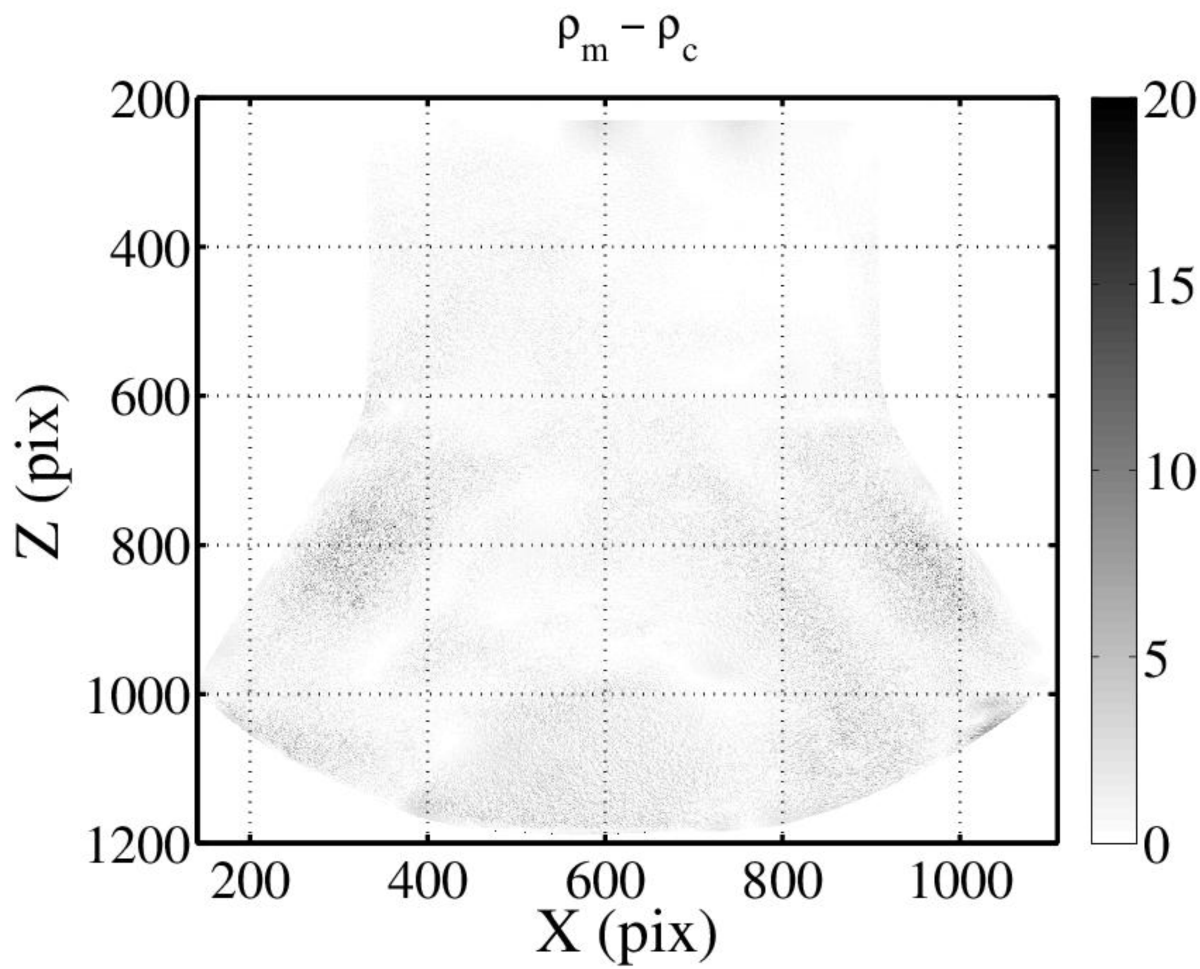
15 unknowns to be identified!

[Gras *et al.*, 2015, *Int. J. Solids Struct.* 55 pp. 2-16]

Residuals

Check for compatibility of identified values with image registration

Assess improvement from single zone description



[Gras *et al.*, 2015, *Int. J. Solids Struct.* 55 pp. 2-16]

DVC FOR 3D IDENTIFICATION

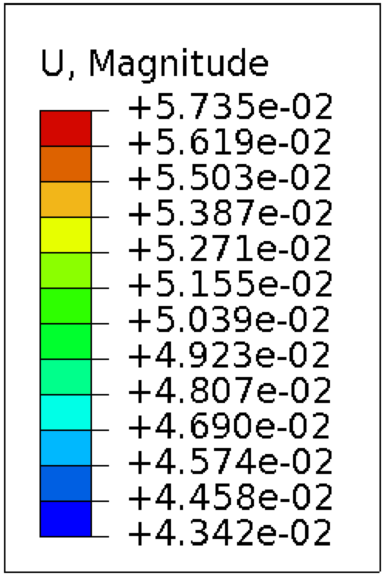
3D Identification

- Indented plaster foam
- Pore collapse criterion?

- DVC based on a set of Abaqus computed displacement fields

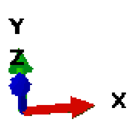
[Bouterf *et al.*, 2014, *Strain* 50 444-453]

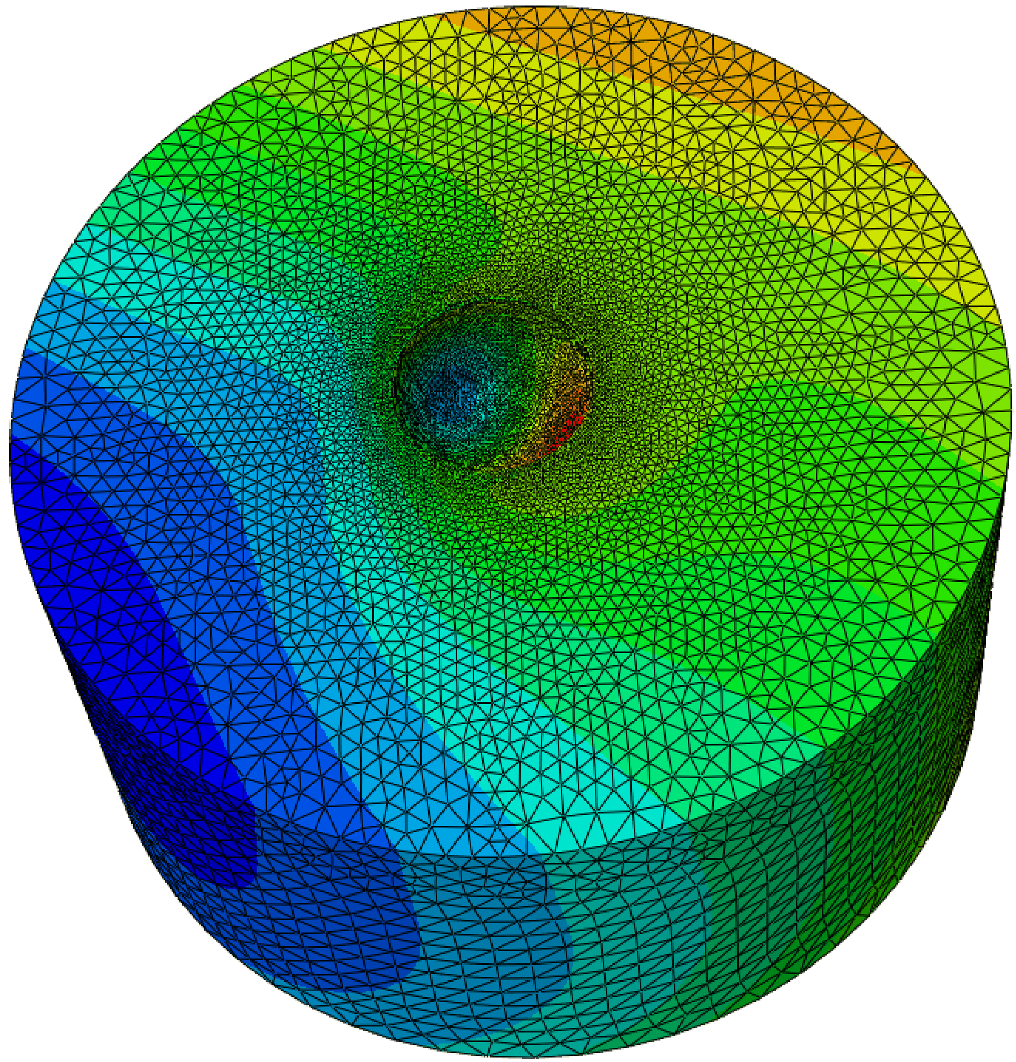
Measured Displacement Field



(mm)

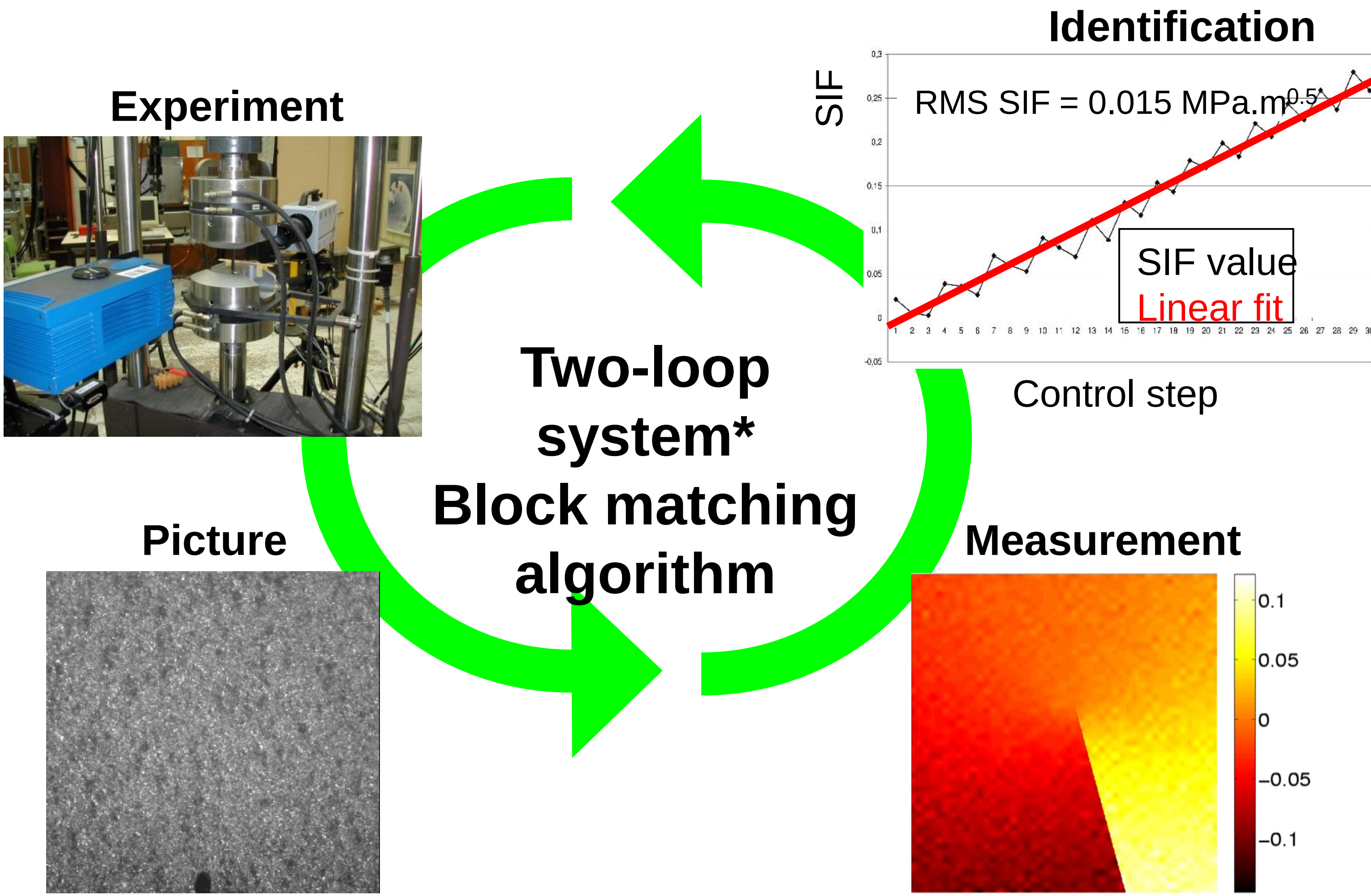
Full displacement range = 14 µm ≈ 1 voxel





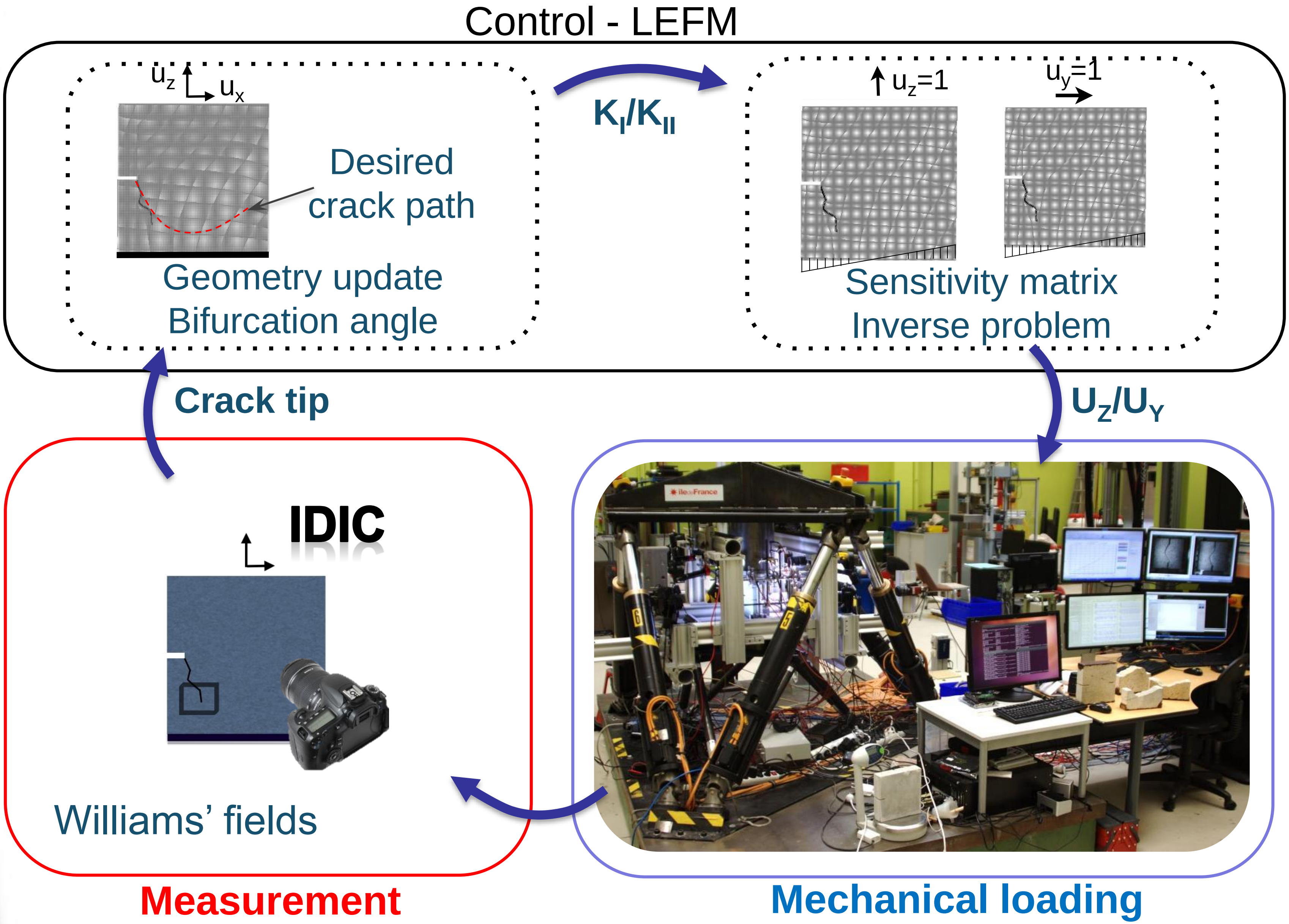
DIC FOR CONTROL

DIC-Controlled Test

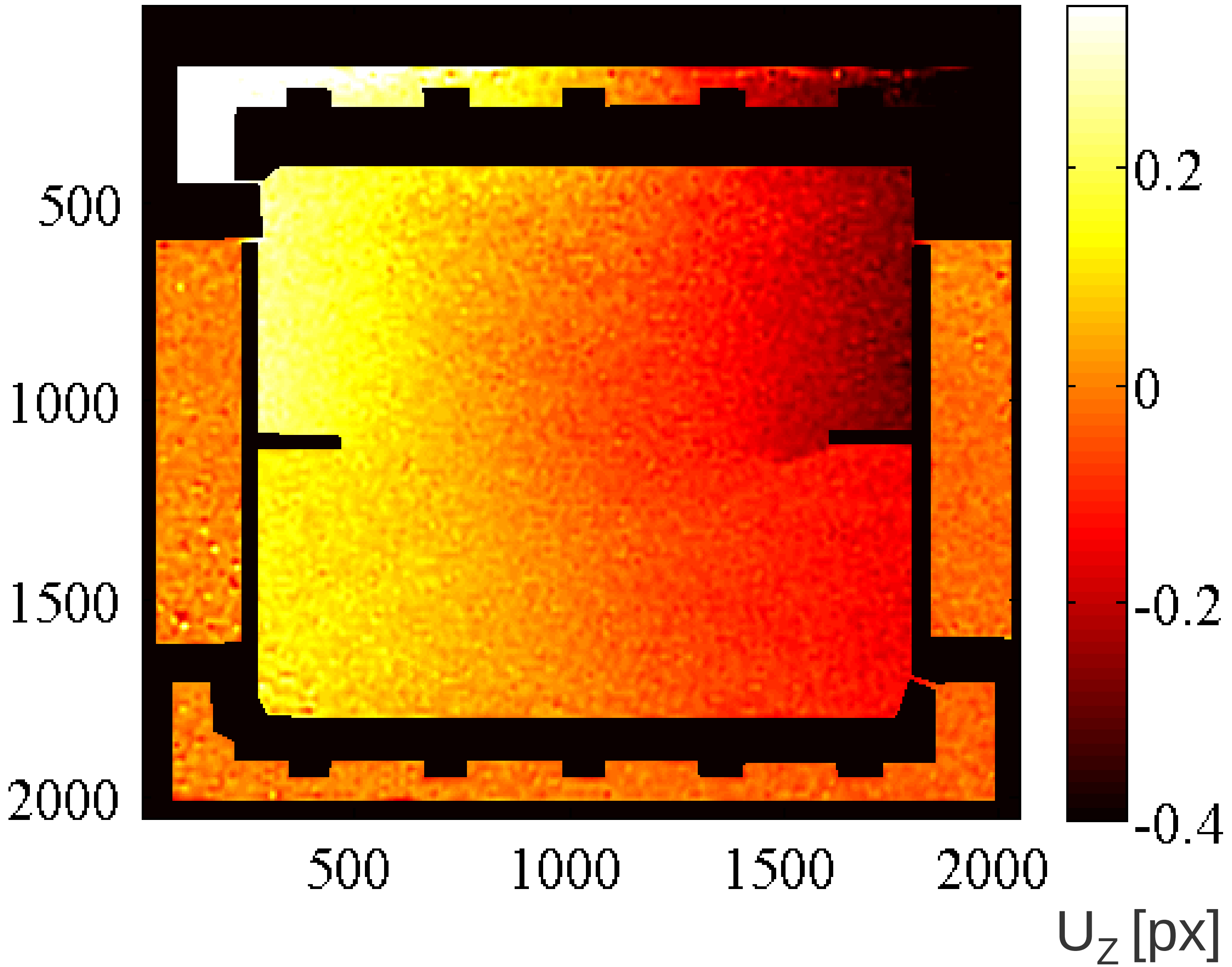


[Fayolle *et al.*, 2007, *Exp. Tech.* 31 pp. 57-63]
[Fayolle *et al.*, 2013, *Exp. Mech.* 54(2) pp. 305-314]

Hybrid Test

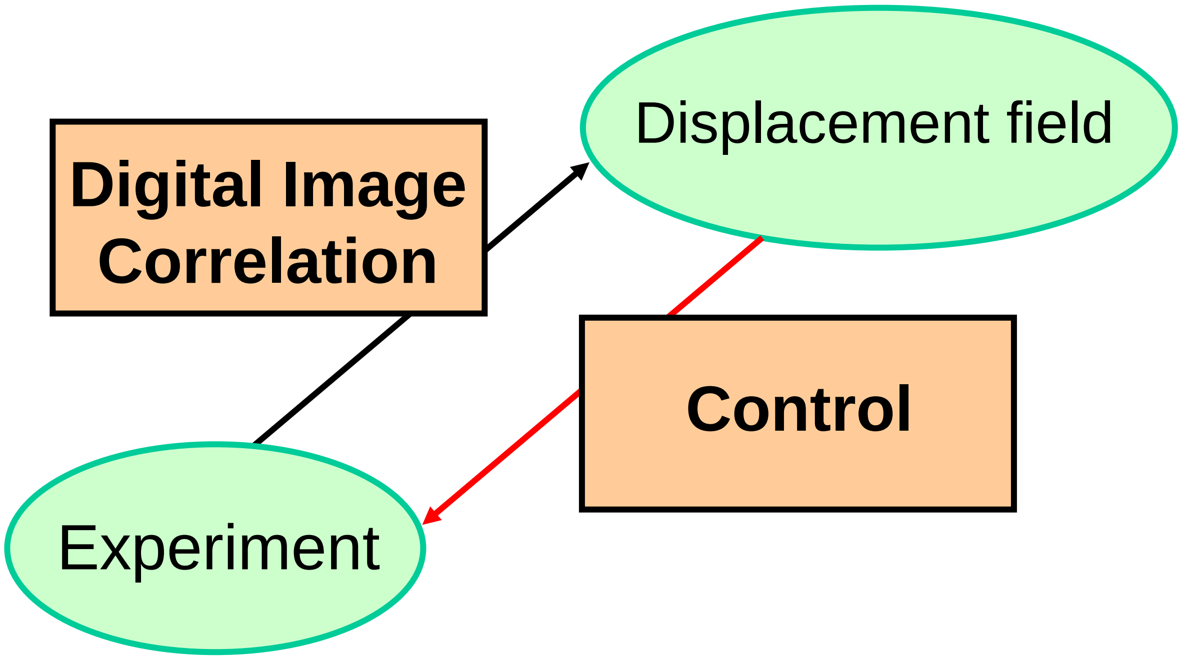

Control - LEFM
u_z u_x
Desired crack path
Geometry update
Bifurcation angle
K_I/K_{II}
$u_z=1$
$u_y=1$
Sensitivity matrix
Inverse problem
Crack tip
U_Z/U_Y
IDIC
Williams' fields
Measurement
Mechanical loading

Hybrid Test



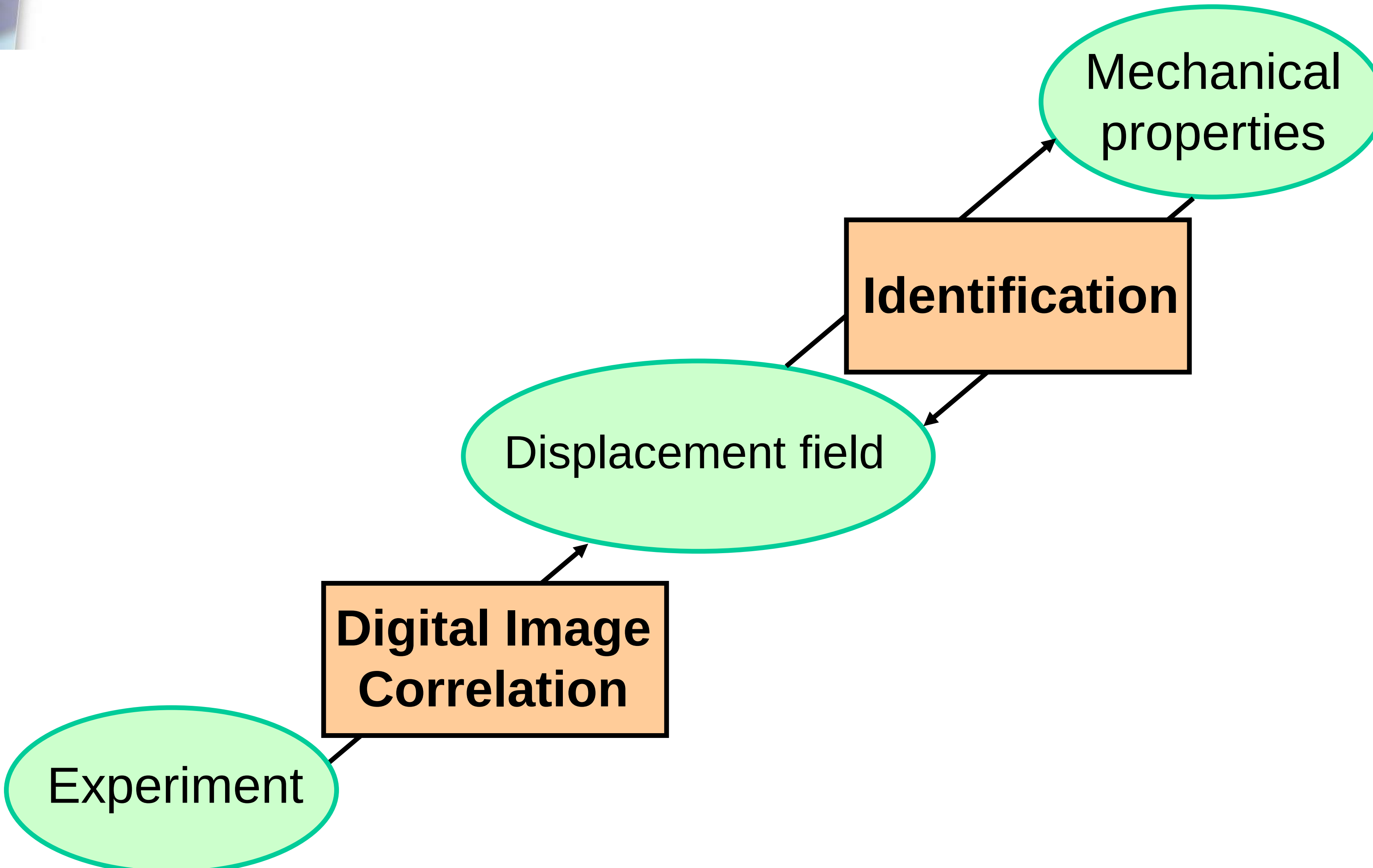
CONCLUSIONS

Summary and Perspectives

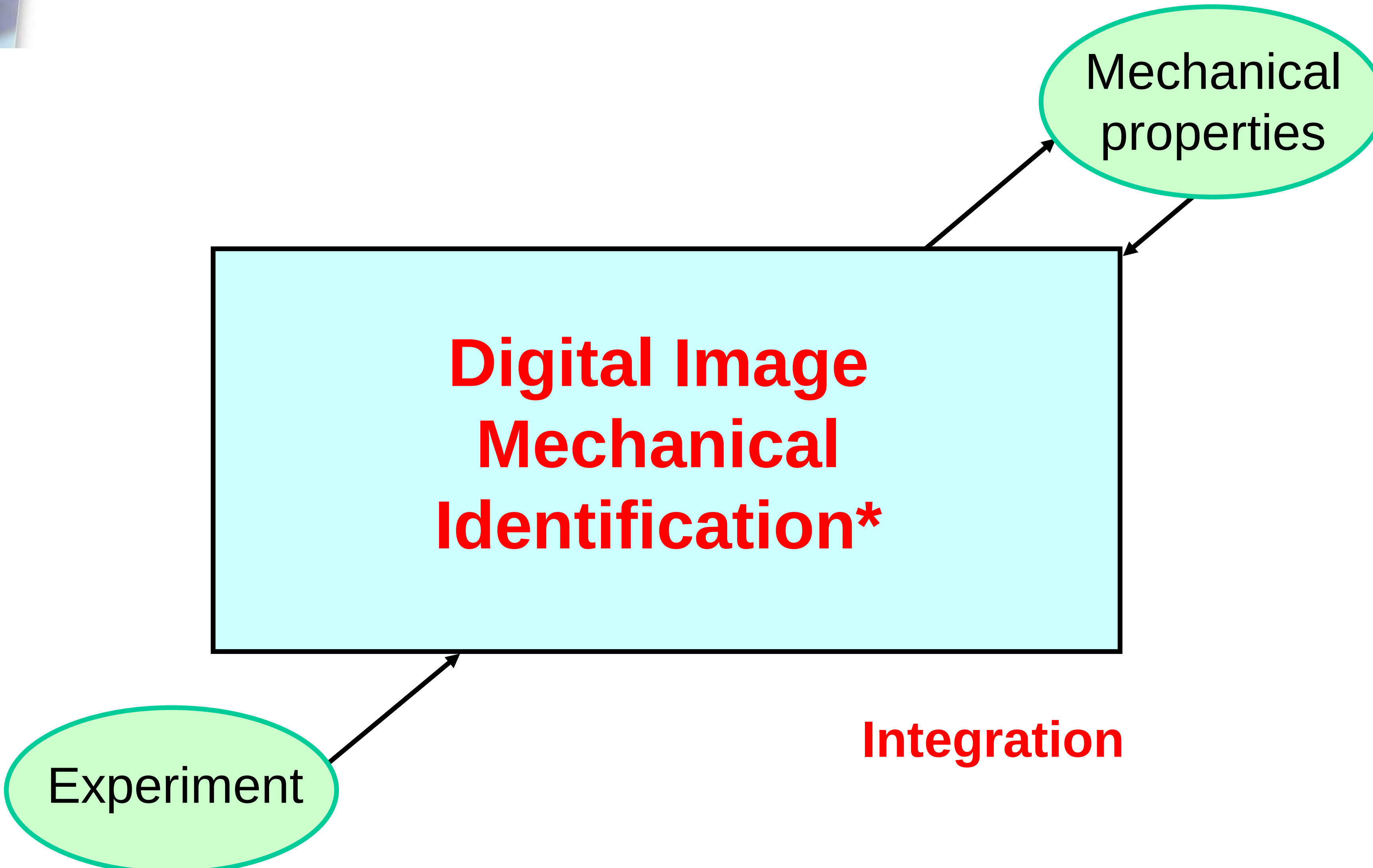


[Fayolle *et al.*, 2007, *Exp. Tech.* 31 pp. 57-63]
[Fayolle *et al.*, 2013, *Exp. Mech.* 54(2) pp. 305-314]

Summary and Perspectives



Summary and Perspectives



*[SR *et al.*, 2008, *Exp. Mech.*, 48, pp. 495-508]

Summary and Perspectives

Smart test

Mechanical properties

Digital Image Mechanical Identification & Control

Experiment

[Bertin *et al.*, 2016, *Strain* 52 pp. 307-323]
[Jailin *et al.*, 2017, *J. Mech. Phys. Solids* 102 239-256]

Why DIC?

- DIC is the perfect interface between experiment and modelling
- It is tolerant to *real* (i.e., “imperfect”) experiments
- It is even friendly with application tests in service conditions
- It can deal with noisy data (image) provided noise is well characterized
- It provides an unprecedented wealth of information provided questions are asked “politely”

Further Reading

- F. Hild, S. Roux, Digital image correlation, *in* ``Optical Methods for Solid Mechanics _ A Full-Field approach'', P. Rastogi and E. Hack eds., Chap. 5, pp. 183-228, Wiley, Berlin, (2012)
- M. Bornert, F. Hild, J.-J. Orteu, S. Roux, Digital Image Correlation, *in* ``Full Field Measurement in Solid Mechanics'', M. Grédiac and F. Hild eds., Chap. 6, pp. 157-190, Wiley-ISTE, London, (2013)
- A. Buljac et al., Digital Volume Correlation: Review of Progress and Challenges, *Experimental Mechanics* 58(5), 661-708, (2018)

- Most research papers from our group are freely accessible from **https://hal.archives-ouvertes.fr/** (the site offers an English version by clicking on the top-right token « en » on the first page)