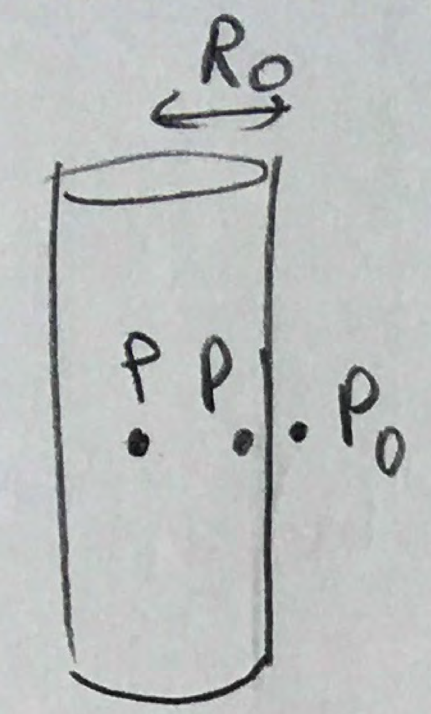
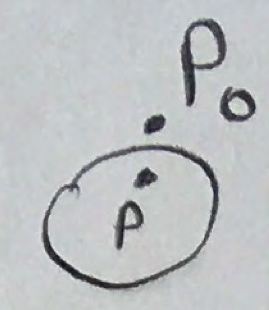
TD3 ex 4 (1)

a)





loi de Laplace :

$$\underbrace{\Delta P}_{>0} = \underbrace{\frac{\gamma}{R_0}}_{>0} = \underbrace{P - P_0}_{>0}$$

car "P pousse l'interface"

$$\Rightarrow P = P_0 + \frac{\gamma}{R_0} \text{ vrai } \forall z.$$

$\Rightarrow \forall z$, proche de l'interface,

$$P = P_0 + \frac{\gamma}{R_0}.$$

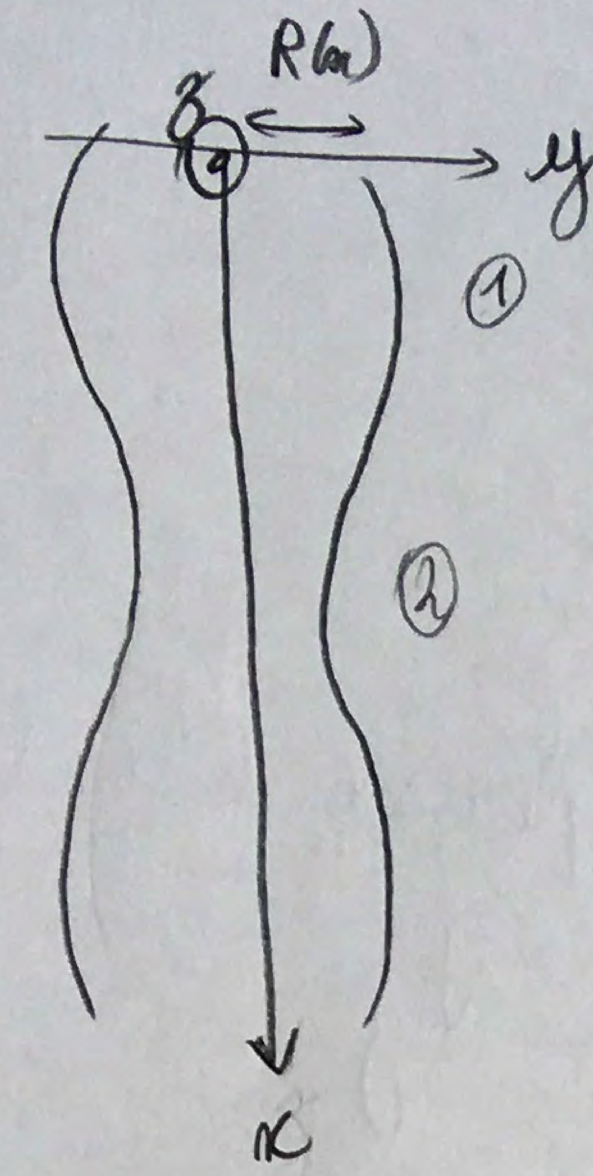
pression dans le reste du cylindre ?

hydrostatique $\Rightarrow P_A - P_B = -\rho g \underbrace{(z_A - z_B)}_{=0 \text{ si } z_A = z_B} = 0 \Rightarrow P_A = P_B$ si $z_A = z_B$.

à z donné, $P = cte$ dans tout le cercle.

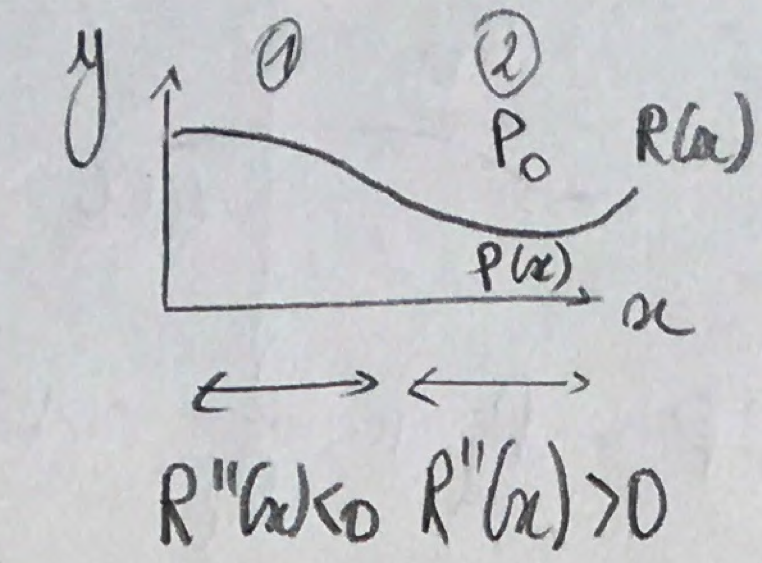
conclusion

dans tout le cylindre $P = P_0 + \dfrac{\gamma}{R_0}$.

c) 

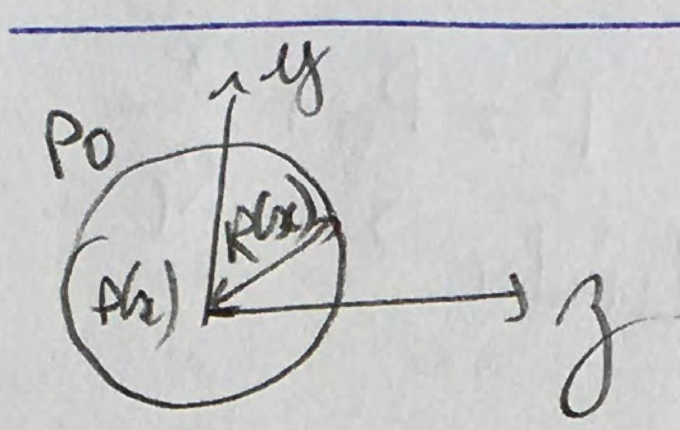
$$R(x) = R_m + \varepsilon \cos(kx)$$

dans le plan (x, y) :



$$(P_0 - P_{(x)})_{xy} = \gamma R''(x).$$

dans le plan (y, z) :



$$(p(x) - P_0)_{yz} = \frac{\gamma}{R(x)}.$$

d'où : $p(x) - p_0 = \dfrac{\gamma}{R(x)} - \gamma R''(x)$

$$= \frac{\gamma}{R_m + \varepsilon \cos(kx)} + \gamma \varepsilon k^2 \cos(kx).$$

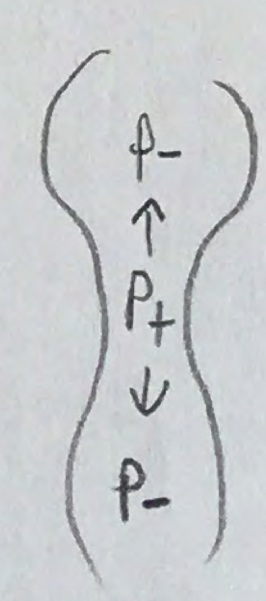
$$= \frac{\gamma}{R_m\left(1 + \underbrace{\frac{\varepsilon}{R_m}}_{\ll 1} \underbrace{\cos(kx)}_{<1}\right)} \sim \frac{\gamma}{R_m}\left(1 - \frac{\varepsilon}{R_m}\cos(kx)\right)$$

$$\frac{1}{1+x} \sim 1 - x.$$

$$p(x) - p_0 = \gamma \varepsilon \cos(kx) \underbrace{\left(k^2 - \frac{1}{R_m^2}\right)}_{<0}$$

① $R''(x) < 0$, $\cos(kx) > 0$

② $R''(x) > 0$, $\cos(kx) < 0$



si $k^2 R_m^2 < 1$,
en ① $p(x) - p_0 < 0 \Rightarrow p-$ (dépress°)
en ② $p(x) - p_0 > 0 \Rightarrow p+$ (surpress°)

+ écoulement va des hautes pressions vers les basses (surpress° pousse, dépress° aspire).

⇒ écoulement depuis les creux vers les bosses ⇒ l'amplitude de la perturbat° augmente ⇒ instabilité

TD3 ex 4 (2)

d) conservation de la masse entre filet non perturbé et filet perturbé.

(ρ = cte $\Rightarrow$ conservation du volume) $\Rightarrow V_i = V_p$.

- filet non perturbé :

$$\boxed{V_i = \pi R_0^2 \lambda}$$

- filet perturbé :

$$V_p = \int_{x=0}^{\lambda} \int_{\theta=0}^{2\pi} \int_{r=0}^{R(x)} r\,d\theta\,dr\,dx$$

$$= 2\pi \int_{x=0}^{\lambda} \frac{R^2(x)}{2}\,dx$$

$$= \pi \int_{x=0}^{\lambda} \left(R_m + \varepsilon \cos(kx)\right)^2 dx$$

$$= \pi R_m^2 \int_{x=0}^{\lambda} \left(1 + 2\frac{\varepsilon}{R_m}\cos(kx) + \frac{\varepsilon^2}{R_m^2}\underbrace{\cos^2(kx)}_{\frac{1+\cos(2kx)}{2}}\right) dx$$

$$= \pi R_m^2 \left(\lambda + \frac{2\varepsilon}{R_m k}\underbrace{\sin(k\lambda)}_{=0 \text{ car } k\lambda = 2\pi} + \frac{\varepsilon^2}{2R_m^2}\lambda + \frac{\varepsilon^2}{4R_m^2 k}\underbrace{\sin(2k\lambda)}_{=0}\right)$$

$$\boxed{V_p = \pi \lambda R_m^2 \left(1 + \frac{\varepsilon^2}{2}\right)}$$

$$V_i = V_p \Rightarrow \cancel{\pi} R_0^2 \cancel{\lambda} = \cancel{\pi}\cancel{\lambda}\left(R_m^2 + \frac{\varepsilon^2}{2}\right)$$

$$\Rightarrow R_m = \sqrt{R_0^2 - \frac{\varepsilon^2}{2}}.$$

$$= R_0 \sqrt{1 - \frac{\varepsilon^2}{2R_0^2}}$$

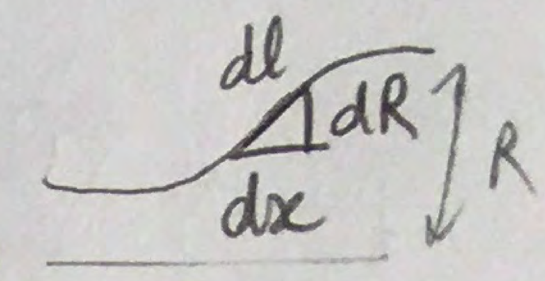
$$\boxed{R_m \sim R_0 \left(1 - \frac{\varepsilon^2}{4R_0^2}\right)}$$

e) TD 3 ex 4 (3)

surface initiale : $\boxed{S_i = 2\pi R_0 \lambda}$

calcul de dl

$dl^2 = dx^2 + dR^2$



$dl = dx\sqrt{1+\left(\frac{dR}{dx}\right)^2}$ $\quad R'(x) = -\varepsilon k \underbrace{\sin(kx)}_{<1}$

$\ll 1$ car $\varepsilon k \ll 1$

$\sim dx\left(1+\frac{1}{2}\left(\frac{dR}{dx}\right)^2\right) \sim dx\left(1+\frac{1}{2}(-\varepsilon k \sin(kx))^2\right)$

$dl \sim dx\left(1+\frac{1}{2}\varepsilon^2 k^2 \sin^2(kx)\right)$

surface perturbée :

$$S_p = \int_{\theta=0}^{2\pi}\int_{l=0}^{l(\lambda)} R(x)\, d\theta\, dl$$

$$= 2\pi \int_{x=0}^{\lambda} R_m\left(1 + \frac{\varepsilon}{R_m}\cos(kx)\right)\left(1+\frac{1}{2}\varepsilon^2 k^2 \sin^2(kx)\right) dx$$

$$= 2\pi R_m \int_{x=0}^{\lambda}\left(1 + \frac{\varepsilon}{R_m}\cos(kx) + \frac{1}{2}\varepsilon^2 k^2 \underbrace{\sin^2(kx)}_{\frac{1-\cos(2kx)}{2}} + \mathcal{O}(3)\right) dx$$

$$= 2\pi R_m\left(\lambda + \frac{\varepsilon}{R_m k}\underbrace{\sin(k\lambda)}_{=0} + \frac{\varepsilon^2 k^2}{4}\lambda - \frac{\varepsilon^2 k^2}{4}\underbrace{\frac{\sin(2k\lambda)}{2k}}_{=0}\right)$$

$$= 2\pi R_m \lambda\left(1 + \frac{\varepsilon^2 k^2}{4}\right)$$

$$\boxed{S_p = 2\pi R_m \lambda\left(1 + \frac{\varepsilon^2 \pi^2}{\lambda^2}\right)}$$

$$S_p - S_0 = 2\pi R_m \lambda\left(1 + \frac{\varepsilon^2\pi^2}{\lambda^2}\right) - 2\pi R_0 \lambda$$

$$= 2\pi\lambda\left(R_m - R_0 + \frac{\varepsilon^2\pi^2}{\lambda^2}R_m\right)$$

$$= 2\pi\lambda R_0\left(\cancel{1} - \frac{\varepsilon^2}{4R_0^2} - \cancel{1} + \frac{\varepsilon^2\pi^2}{\lambda^2}\left(1 - \frac{\varepsilon^2}{4R_0^2}\right)\right)$$

$$= 2\pi\lambda R_0\left(-\frac{\varepsilon^2}{4R_0^2} + \frac{\varepsilon^2\pi^2}{\lambda^2} + \mathcal{O}(4)\right)$$

$$S_p - S_0 < 0 \Rightarrow \frac{\varepsilon^2 \pi^2}{\lambda^2} < \frac{\varepsilon^2}{4 R_0^2}$$

$$\Rightarrow \lambda^2 > 4\pi^2 R_0^2 .$$

$$\Rightarrow \lambda > 2\pi R_0$$

$$\Rightarrow \frac{\lambda}{2R_0} > \pi \in [3,13 ; 3,18] \longrightarrow \text{en accord avec Plateau}$$

si $S_p - S_0 < 0$, $S_p < S_0$ donc pour minimiser l'énergie de surface, le système favorisera l'état perturbé