chap 2: Ergodicity and counting states

I) Reminder of classical mechanics

1) Single particle

Nat° of classical particle of mass m: $\vec{F} = m\vec{a}$

$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}}$ accelerat°

$\vec{F}$ = forces due to other particles, external forces (gravit°, electric, magnetic, etc...)

Set $\vec{p} = m\vec{v}$

$$\Rightarrow \begin{cases} \dot{\vec{r}}(t) = \dfrac{\vec{p}(t)}{m} \\ \dot{\vec{p}}(t) = \vec{F}(\vec{r}(t), \vec{p}(t), t) \end{cases}$$

$\Rightarrow$ Linear in time eqs. $\Rightarrow$ If we know $\vec{r}(t_0) = \vec{r}_0$; $\vec{p}(t_0) = \vec{p}_0$ we can integrate them.

$$\begin{cases} \vec{r}(t_0 + \Delta t) = \vec{r}_0 + \Delta t \dfrac{\vec{p}(t_0)}{m} + O((\Delta t)^2) \\ \vec{p}(t_0 + \Delta t) = \vec{p}_0 + \Delta t\, \vec{F}(\vec{r}(t_0), \vec{p}(t_0), t_0) + O((\Delta t)^2) \end{cases}$$

For infinitesimal Δt, one can iterate this integrat° procedure and thus find $(\vec{r}(t), \vec{p}(t))$ for all $t \geq t_0$

Solut° defines a curve or trajectory in the space $\{\vec{r}, \vec{p}\}$ parametrized by t

For a single particle, the space $\{\vec{r}, \vec{p}\}$ has dim 6

This space is called <u>phase space</u>

2) Extension to N particles

Newton law: $\vec{F}_i = m\vec{a}_i$; $i \in [\![1, \dots, N]\!]$ same mass

Again $\vec{a}_i = \dot{\vec{v}}_i = \ddot{\vec{r}}_i$

Assume only Forces due to external potential + interact between particles

Ext. force could be a gravitational force

Int. pot: Lennard Jones we already met before

Energy of the system: $\mathcal{H} = \mathcal{H}(\vec{r}_1, \dots, \vec{r}_N, \vec{p}_1, \dots, \vec{p}_N)$

$= E_{kin} + E_{pot}$

where $E_{kin} = \sum_{i=1}^{N} \frac{p_i^2}{2m}$; $E_{pot} = \sum_{i=1}^{N} V_{ext}(\vec{r}_i) + \sum_{1 \le i < j \le N} V_{int}(|\vec{r}_i - \vec{r}_j|)$

Then $\vec{F}_i = -\frac{\partial E_{pot}}{\partial \vec{r}_i} = -\vec{\nabla}_{r_i} E_{pot} = -\overrightarrow{grad}_i E_{pot}$

or $\partial_{\vec{r}_i} = \left(\frac{\partial}{\partial r_{ix}}, \frac{\partial}{\partial r_{iy}}, \frac{\partial}{\partial r_{iz}}\right)$

Therefore, the equations of motion for our set of N particles are

$$\begin{cases} \dot{\vec{r}}_i = \dfrac{\vec{p}_i}{m} \\ \dot{\vec{p}}_i = -\dfrac{\partial}{\partial \vec{r}_i} E_{pot}(\vec{r}_1(t), \vec{r}_2(t), \dots, \vec{r}_N(t)) \end{cases}$$

6N eqs
first order

$$\Leftrightarrow \begin{cases} \dot{\vec{r}}_i = \dfrac{\partial \mathcal{H}}{\partial \vec{p}_i} \\ \dot{\vec{p}}_i = -\dfrac{\partial \mathcal{H}}{\partial \vec{r}_i} \end{cases}$$

called Hamilton-Jacobi eqs

$\mathcal{H}$ = Hamiltonian of the system.

$\rightsquigarrow$ main interest of this more abstract formalism: we can extend it to a set of conjugate variables $\{\vec{r}_i, \vec{p}_i\} \to \{\theta_i, \frac{\partial}{\partial \theta_i}\}$

Conjugate here means in the sense of classical mechanics (action---)

RQ: In classical mechanics $(\vec{q}_i, \vec{p}_i)$ instead of $(\vec{r}_i, p_i)$

Define $\vec{\vec{X}} = (\vec{r}_1, \dots, \vec{r}_N)$; $\vec{\vec{P}} = (\vec{p}_1, \dots, \vec{p}_N)$

$\vec{\Gamma}(t) = (\vec{X}(t), \vec{P}(t))$ is a trajectory in a 6N-dimension space.

trajectory is fully determined.

no crossing!

RQ!: If $\mathcal{H}$ is indept of t, energy is conserved $\Rightarrow$ one constraint $H(\vec{X}, \vec{P}) = E$

Ex: Prove $\frac{d\mathcal{H}}{dt} = 0$

Rq: Other conserved quantities are possible (such as total momentum, angular momentum, etc...)

As already emphasized, we cannot perform the $6N$ dif.-integration
- Not possible with computer
- We do not know the initial conditions and we can never know

This is not a major pb since the macroscopic behaviour is often described by a small set of variables ($E, T, p, \ldots$)
- Most macroscopic systems tend toward a eq. (stationary) regime independent of initial condit°
- The measure of an observable happens on a time scale long enough compare to the microscopic time of evolut°

This implies that we necessary average over configurat° in phase space ⇒ Statistical descript° is necessary in our macros. world

II) Macrostates

2.1) Definition

Def: microstate = point in phase space. ($6N$ dimension)

Since we never know the microstate of a system, we know pass from simple mechanics to stat. mechanics

$\rho(\vec{\Gamma})$ = probability density. ||

Therefore $P[\vec{\Gamma}] \equiv \rho(\vec{\Gamma})\, d^{6N}\vec{\Gamma}$ ≡ proba of finding the system in the elementary volume $d^{6N}\vec{\Gamma}$ around $\vec{\Gamma}$

obviously $\int \rho(\vec{\Gamma})\, d^{6N}\Gamma = 1$

$\Leftrightarrow \int \rho(\vec{\Gamma})\, d\vec{r}_1 \cdots d\vec{r}_N\, d\vec{p}_1 \cdots d\vec{p}_N = 1$

Def: A macrostate is thus defined by $\rho(\vec{\Gamma})$ ||

objective: Calculate $\rho(\vec{\Gamma})$ which, as we will see, depends on a small set of parameters

Consider an observable O (pressure of a fluid)

$$\langle O(t) \rangle = \int d^{6N}\Gamma \; \rho(\vec{\Gamma}, t)\, \mathcal{O}(\vec{\Gamma})$$

we will come back to this definit° and its relat° to a phys. observable in a genuine experiment.

2.2 Flow in phase space and Liouville eq.

We have seen that a given point in phase space described a trajectory in phase space. However, statistically a system at time t_0 is described by a density $\rho(\vec{\Gamma}, t_0)$ composed of an ensemble of microstates. At time t_1 we have a new set of microstates $\Rightarrow \rho(\vec{\Gamma}, t_1)$

Q: how do we determine/relate $\rho(\Gamma, t)$ using Hamilton-Jacobi equations!

$\frac{D\rho}{Dt} \equiv \frac{d}{dt}\rho(\vec{\Gamma}(t); t)$ ← change of ρ at fixed $\vec{\Gamma}$ with t

$= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \vec{\Gamma}} \frac{\partial \vec{\Gamma}}{\partial t}$ ← change of ρ at fixed t, but $\vec{\Gamma}$ moved

$= \frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial \rho}{\partial \vec{p}_i} \frac{\partial \vec{p}_i}{\partial t} + \frac{\partial \rho}{\partial \vec{q}_i} \frac{\partial \vec{q}_i}{\partial t} \right)$

$= \frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial \rho}{\partial \vec{p}_i} \left(-\frac{\partial H}{\partial \vec{q}_i} \right) + \frac{\partial \rho}{\partial \vec{q}_i} \left(\frac{\partial H}{\partial \vec{p}_i} \right) \right)$

$= \frac{\partial \rho}{\partial t} + \{\rho, H\}$ where $\{A, B\} = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial \vec{q}_i} \frac{\partial B}{\partial \vec{p}_i} - \frac{\partial A}{\partial \vec{p}_i} \frac{\partial B}{\partial \vec{q}_i} \right)$

↑ Poisson brackets

we next need to calculate $\frac{\partial \rho}{\partial t}$

The idea is to use the fact that every point in phase space present at time t_0 will be present at time t_{on} → no new points appear or disappear.

It follows that $\rho(\vec{\Gamma}(t), t)$ follows a continuity equation

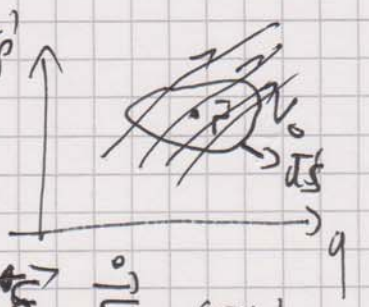
Therefore $\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{V}) = 0$

here $\vec{V}$ is a $6N$-dimensional velocity in phase space.

Basically $\vec{V} \equiv \dot{\vec{\Gamma}}$

This is the local $\subseteq$ eq for ρ. Its global integral form is defined/obtained as follows: $P_{\vec{\Gamma}} = \rho(\vec{\Gamma})\, d^{6N}\vec{\Gamma}$ proba of finding the system within the volume $d^{6N}\Gamma$ around $\vec{\Gamma}$

Consider a volume (fixed) V_0 around $\vec{\Gamma}$



$$P_{V_0}(\vec{\Gamma}) = \int_{V_0} d^{6N}\Gamma \; \rho(\vec{\Gamma},t)$$

$$\frac{d P_{V_0}(\vec{\Gamma})}{dt} = \int_{V_0} d^{6N}\Gamma \; \frac{\partial \rho}{\partial t}(\vec{\Gamma},t) = -\iint_{S_0} d\vec{S}\cdot\dot{\vec{\Gamma}}\,\rho(\Gamma,t)$$

$$= -\iiint_{V_0} d^{6N}\Gamma \; \vec{\nabla}\cdot\left(\dot{\vec{\Gamma}}\,\rho(\vec{\Gamma},t)\right) \quad \forall\, V_0$$

Green-Ostrogradski $\iiint_V \vec{\nabla}\cdot\vec{F}\, dV = \iint_{\partial V} \vec{F}\cdot d\vec{S}$

$\Rightarrow$ Thus $\frac{\partial \rho(\vec{\Gamma},t)}{\partial t} + \vec{\nabla}\cdot\left(\dot{\vec{\Gamma}}\,\rho(\vec{\Gamma},t)\right) = 0$ ($\subseteq$ equation)

$\vec{\nabla}\cdot\left(\dot{\vec{\Gamma}}\,\rho(\Gamma,t)\right)$ = div$\left(\dot{\vec{\Gamma}}\,\rho(\Gamma,t)\right)$

$$\vec{\nabla}\cdot\left(\dot{\vec{\Gamma}}\,\rho(\Gamma,t)\right) = \dot{\vec{\Gamma}}\cdot\vec{\nabla}\rho(\vec{\Gamma},t) + \rho(\vec{\Gamma},t)\,\vec{\nabla}\cdot\dot{\vec{\Gamma}}$$

and $\vec{\nabla}\cdot\dot{\vec{\Gamma}} = \sum_{i=1}^{N}\left(\frac{\partial}{\partial\vec{p}_i}\dot{\vec{p}}_i + \frac{\partial}{\partial\vec{q}_i}\dot{\vec{q}}_i\right) = \sum_{i=1}^{N} -\frac{\partial}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial}{\partial q_i}\frac{\partial H}{\partial p_i} =$

$$= \sum_{i=1}^{N} -\frac{\partial^2 H}{\partial p_i \partial q_i} + \frac{\partial^2 H}{\partial q_i \partial p_i} = 0$$

$$\vec{\nabla}\cdot\left(\dot{\vec{\Gamma}}\,\rho\right) = \dot{\vec{\Gamma}}\cdot\nabla\rho(\Gamma,t) = \sum_{i=1}^{N}\left(\dot{\vec{p}}_i\frac{\partial\rho}{\partial\vec{p}_i} + \dot{\vec{q}}_i\frac{\partial\rho}{\partial q_i}\right) = \{\rho, H\}$$

$$\Rightarrow \boxed{\frac{\partial\rho(\vec{\Gamma},t)}{\partial t} = -\{\rho, H\}} \quad \text{Liouville eq.}$$

$$\Rightarrow \boxed{\frac{d\,\rho(\vec{\Gamma}(t),t)}{dt} = 0} \quad \text{Liouville } \underline{\text{thm}} \text{ conservation of density}$$

equilibrium $\underline{\rho = \rho_{stat}} = \rho(\vec{\Gamma})$

$= f(\underbrace{H(\vec{\Gamma})}_{\text{Hamiltonian}})$

In general $\rho(\vec{r}, t)$ depends on time.

Therefore any average depends on time

* However, we know experimentally that an isolated system will tend to an equilibrium state $\rho(\vec{r}, t) \longrightarrow \rho(\vec{r})$

=> The point is that these ensembles do depend on a few parameters

They are not hard to find.

We will discuss three typical ensembles:

- ensemble microcanonical → extensive variables
  → isolated system ($E, V, N, \dots$ fixed.)
- canonical ensemble
  The system is in contact with a thermal reservoir
  ~ T is fixed (E is no longer conserved, heat exchange)
- grand-canonical ensemble
  The system is in contact with a reservoir and can exchange with it, heat and particles
  (E and N are no longer conserved)
  However μ and T are imposed.

~ Based on the number of constraints imposed on the system under considerations, other ensembles can be envisioned. (see TD)

RQ: In stationary ensembles, the partition function

$$Z = \int d\vec{r}\ \rho(\vec{r})$$ will play a key role

At this stage, this is simply a normalization but once the ensemble $\rho(\vec{r})$ are determined, it will be used as a generating function →

RQ: extensive variables

A | A' (double of A)

$E_{2A} = 2E_A$; $N_{2A} = 2N_A$, $V_{2A} = 2V_A$, Ext

However T_{2A}, S_{2A}, P_{2A} remain the same → Intensive

Remarque Extensive versus intensive variables

N = nb of particles

Set X_i = some other extensive variables

Then $x_i = \frac{X_i}{N}$ is intensive

Let Y another extensive variable $= Y(N, X_1, X_2, \dots)$

Then $y = \frac{Y}{N} = \frac{1}{N} Y(N, X_1, X_2, \dots)$ is intensive

$= y(x_1, x_2, \dots)$

An intensive variable, say y, can only depend on ratios of extensive variables, here $\frac{X_i}{N} = x_i \Rightarrow$ reduced variables

These notions will be important later to define what we call the thermodynamic limit

Thermodynamics applies when the size of the system we study is much larger than any other microscopic length scales in the system (length scale associated with exchange of heat, etc.)

// Mathematically, we send $N \to \infty$ keeping all intensive variables fixed

2.3] Ergodicity

Microstate = point in phase space

Macrostate = proba density in phase space : $\rho(\vec{\Gamma})$

$\rho(\vec{\Gamma})$ depends on a small nb of variables

⇒ Sinai problem : trajectory of a sphere in a billard.
very simple pb to model the trajectory of a molecule in presence of the others

We numerically learnt that the phase space which is accessible is uniformly visited provided we wait an enough long time!

Consider a physical observable $\mathcal{O}(\{\vec{q}_i, \vec{p}_i\}) \equiv \mathcal{O}(\vec{\Gamma})$

Define • $\overline{\mathcal{O}_T} = \frac{1}{T} \int_{t_0}^{t_0+T} dt\ \mathcal{O}(\{\vec{\Gamma}(t)\})$ (1)

a time average

for $T \gg \{t_i\}$ ↳ any microscopic time

• $\langle \mathcal{O} \rangle = \int \prod_i dq_i dp_i\ \rho(\{\vec{q}_i, \vec{p}_i\})\ \mathcal{O}(\{\vec{q}_i, \vec{p}_i\})$ (2)

↑ statiscal average in phase space

A priori $\overline{\mathcal{O}_T}$ and $\langle \mathcal{O} \rangle$ are different

The 1st one, $\overline{\mathcal{O}_T}$, is difficult (impossible) to calculate.

For the 2nd one, we need $\rho(\vec{\Gamma})$ → doable and much easier.

Ergodic hypothesis : $\lim_{T \to \infty} \overline{\mathcal{O}_T} = \langle \mathcal{O} \rangle$

- valid in most cases (especially in every day life)
- Breaks down for mesoscopic systems
- A word about many body localization

3) Quantum systems

3.1) Reminder (see your QM course)

In certain situat°, we need a quantum description

- A state of a system is described by a ket $|\psi\rangle \in \mathcal{H}$ → Hilbert space.
- Scalar product $\langle\phi|\psi\rangle$ amplitude of state $|\psi\rangle$ to be in $|\phi\rangle$
- Probabilistic interpretation $P(\psi \to \phi) = |\langle\phi|\psi\rangle|^2$

  $P(r \to \psi) = |\langle r|\psi\rangle|^2$ → occupat° proba of $|\psi\rangle$ at posit° $\vec{r}$
- Schroedinger eq. $H|\psi\rangle = i\hbar \frac{\partial}{\partial t}|\psi\rangle$

  L, linear, reversible

3.2) Micro & Macro state

A Q. state is characterized by "quantum nbs" which correspond to values taken by certain observables.

Ex: Free particle in a box: $\hat{H} = \frac{\hat{p}^2}{2m}$

Wave-particle duality: $\vec{k} = \vec{p}/\hbar$

$\psi(\vec{r}) = \langle r|\psi\rangle = \exp(i\vec{k}\cdot\vec{r})$ plane wave.

we need to satisfy OPC → $\psi(\vec{r})|_{edge} = 0$ → $\psi(\vec{r}) \propto \sin(k_x x)\sin(k_y y)\sin(k_z z)$

with $k_\alpha = \frac{\pi n_\alpha}{L}$ → $E = (n_x^2 + n_y^2 + n_z^2)\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$, δE → level spacing

A quantum state is characterized by a set $\{n_x, n_y, n_z\} \equiv \{\ell\}$

and $\{|\psi_\ell\rangle\} \equiv \mathcal{H}$ Hilbert space.

Microstate ⟷ $\|\psi_\ell\rangle \in \mathcal{H}$

Typically, for a (macroscopic) quantum system, we only know at best the proba for the system S to be in the state $|\psi_\ell\rangle$ → say P_ℓ

Macrostate = $\{P_\ell\} + \{|\psi_\ell\rangle\}$

⇒ statistical mixture

$$\text{Proba}\left\{\phi \leftarrow \text{Macrostate state}\left\{P_\ell, |\psi_\ell\rangle\right\}\right\} = \sum_\ell P_\ell \, |\langle\phi|\psi_\ell\rangle|^2$$

statistical average.

- Consider an observable $\hat{O}$
  - Average value of $\hat{O}$ in the pure state $|\psi\rangle = \langle\hat{O}\rangle_\psi = \langle\psi|\hat{O}|\psi\rangle$
    $\langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2 \neq 0$ Quantum uncertainty.

- For a macro state = stat. mixture.

$$\text{stat} \rightarrow \overline{\langle O\rangle} = \sum_n \underbrace{P_n}_{\text{stat. mean}} \underbrace{\langle\psi_n|\hat{O}|\psi_n\rangle}_{\text{quantum average}}$$

ψ_n

Rq: We can formalize the analogy and define a density operator.

$$\hat{\rho}_\psi = \sum_n P_n |\psi_n\rangle\langle\psi_n|$$

↳ see my course on advance Q.M

4] Counting the micro-states : density of states

4.1] Intro & Motivation

We introduced previously $\rho(\vec{r})$ probability density in phase space.
Or equivalently, we can count the nb of microstate within a macrostate for a quantum system → discrete.
As we will see in many example, the occupation probability of a microstate depends on the energy (see our previous example)
Therefore, we will need to count the number of microstates with a given energy E
⇒ New concept : density of state $g(E)$ or $\nu(E)$

4.2] Definitions

In many cases, we will need to calculate $\sum_\ell f(E_\ell)$ where f regular function
Despite ℓ is discrete, the spectrum of a macrosystem is dense.

$$\sum_\ell f(E_\ell) = \int dE\, g(E)\, f(E)$$

$g(E)$ is the d.o.s : $g(E)\cdot dE$ = nb of microstates in $[E, E+dE]$

or equivalently $g(E) = \sum_\ell \underbrace{\delta(E-E_\ell)}_{\text{Dirac distribution}} \sim 1/\text{Energy}$

$$\int dE \sum_\ell \delta(E-E_\ell) f(E) = \sum_{\ell \leftarrow \text{all q-states}} f(E_\ell)$$

⚠ Do not mix quantum states with energy levels E_ℓ

In general, an energy level E_ℓ has a large degeneracy g_ℓ

$$g(E) = \sum_\ell \delta(E-E_\ell) = \sum_{(E_\ell)\neq\ell} g_\ell\, \delta(E-E_\ell)$$

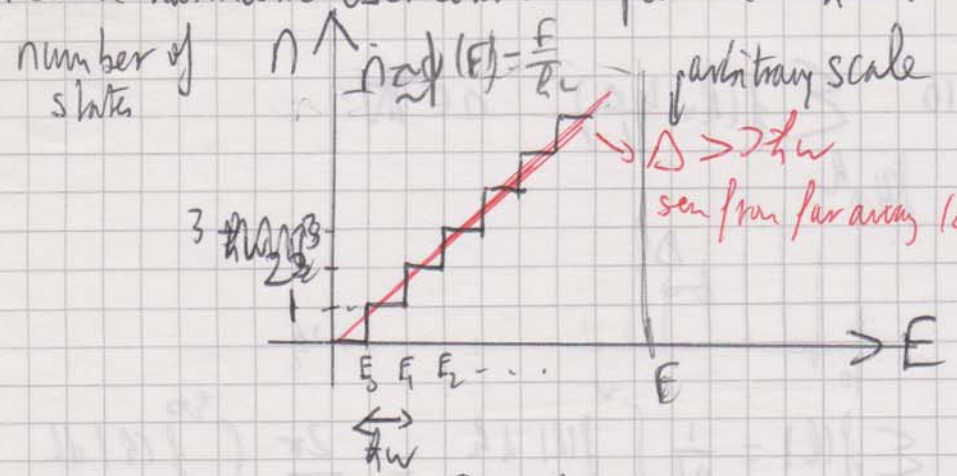
Pb : $g(E)$ is a highly singular object (sum of Dirac peaks)

Integrated density of states : $\phi(E) \equiv \int^E dE'\, g(E')$

$$= \sum_\ell \theta(E-E_\ell) \rightarrow \text{Dimensionless}$$

More regular distribution

For an harmonic oscillator: spectrum $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$



$$\sum_{E_\ell} f(E_\ell) \simeq \int \frac{dE}{\hbar\omega} f(E)$$ if f varies slowly at the scale of $\hbar\omega$

large $E \gg \hbar\omega$ ⇒ $n \simeq \frac{E}{\hbar\omega}$ ← basically $\phi(E) \simeq \frac{E}{\hbar\omega} \equiv \phi_{Weyl}$

ϕ_{Weyl} is "regular" smoother version of $\phi(E)$ or coarse grained.

$$\rho_{Weyl} = \frac{1}{\hbar\omega}$$

At this level, this looks like a mathematical approximation. However, as we will see, the key pt is that the calculation of $\phi_{Weyl} \simeq \phi(E)$ can be formulated in purely classical terms
→ no precise knowledge of quantum spectrum

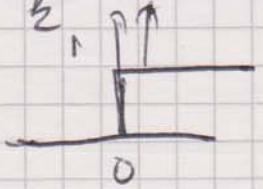
4.3) Density of states in the semi-classical approximation

a) Free particle on a ring of length L i.e.

$H = \frac{p^2}{2m}$ eigenfunction $\psi_k(x) = \frac{e}{\sqrt{L}}$ ⇒ $p = \hbar k$

Periodic b.c $kL = 2\pi n$ ⇒ $k = \frac{2\pi n}{L}$ ⇒ $E = \frac{\hbar^2 k^2}{2m}$

$$\phi(E) = \sum_k \Theta_H\left(E - \frac{\hbar^2 k^2}{2m}\right) = \sum_{n\in\mathbb{Z}} \Theta_H\left[E - \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 n^2\right]$$



If we consider a particle in of box in dim d. (volume $= L^d$)

now $\vec{k} = \frac{2\pi}{L}(n_1, \dots, n_d)$ with $n_i \in \mathbb{Z}$

$$\phi(E) = \sum_{\vec{k}} \Theta_H\left(E - \frac{\hbar^2 \vec{k}^2}{2m}\right)$$

where $\sum_{\vec{k}} = \sum_{n_1 \in \mathbb{Z}} \cdots \sum_{n_d \in \mathbb{Z}}$

The integrated density of states counts all k inside an hypersphere of radius $\sqrt{\frac{2mE}{\hbar^2}} = \frac{\sqrt{2mE}}{\hbar}$

If the volume is large i.e. if $E >> \frac{\hbar^2}{mL^2}$, we can replace the sums by $\int$

(An estimate of $\phi(E)$ is volume of hypersphere / $\left(\frac{2\pi}{L}\right)^d$)

See side page

$$\boxed{\sum_{\vec{k}} \longrightarrow \frac{V}{(2\pi)^d} \int d\vec{k}} \Leftrightarrow \int \frac{dk}{(2\pi)^d} \longleftrightarrow \frac{1}{V} \sum_k$$

Thus $\phi(E) \approx V \int \frac{d^d\vec{k}}{(2\pi)^d} \Theta_H\left(E - \frac{\hbar^2 k^2}{2m}\right)$

$$= \int \frac{d^d\Pi}{} \int \frac{d^d k}{(2\pi)^d} \Theta_H(\quad)$$

$$= \frac{1}{(\hbar 2\pi)^d} \int d^d q \int d^d p \; \Theta_H\left(E - \frac{p^2}{2m}\right)$$

classical phase space

Therefore $\phi(E) = \frac{1}{\hbar^d} \times$ (volume of phase space occupied by the microstates of energy $< E$) ///

Therefore, at large energies, we calculate classicaly the phase space available and divide by $\hbar^d$.

$\hbar^d \approx$ volume occupied by a quantum state in the cl. phase space

$\Rightarrow$ Reminiscent of $\Delta q_x \Delta p_x \geq \hbar/2$

$\Rightarrow$ d = nb of δ^o of freedom.

$$\int d^d r \int d^d p\ \Theta_H\left(E - \frac{p^2}{2m}\right) = V \times (\text{volume of hypersphere of radius } \sqrt{2mE})$$

vol of hsphere of $R=1$: $V_d = \dfrac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$ $\quad \left(\Gamma(x+1) = x\,\Gamma(x)\right)$

$$\Rightarrow \phi(E) = \frac{1}{h^d} V \cdot \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} (2mE)^{d/2} = \frac{V}{\Gamma\left(\frac{d}{2}+1\right)} \left(\frac{mE}{2\pi\hbar^2}\right)^{d/2} = \frac{1}{\Gamma\left(\frac{d}{2}+1\right)} \left(\frac{E}{\delta\varepsilon}\right)^{d/2}$$

where $\delta\varepsilon = \dfrac{2\pi^2\hbar^2}{mL^2}$ (level spacing with PBC)

Ex 2: 1D Harmonic oscillator. $\left(H = \dfrac{\vec{p}^2}{2m} + \dfrac{1}{2} m\omega^2 q^2\right)$

$\int dq\,dp\ \Theta_H(E-H)$ = vol. of ellipse with $\begin{cases} a = \sqrt{2mE} \\ b = \sqrt{\frac{2E}{m\omega^2}} \end{cases}$

$$= \pi \frac{2E}{\omega} \Rightarrow \phi_{\text{weyl}} \approx \frac{2\pi E}{\hbar\omega} = \frac{E}{\hbar\omega} \quad \Rightarrow \text{we recover previous result}$$

$$\left(\rho(E) = \frac{1}{\hbar\omega}\right)$$

4.4) Density of states for identical particles.

here $H = \sum_{i=1}^{N} \left\{ \dfrac{\vec{p}_i^{\,2}}{2m} + U(\vec{q}_1, \ldots, \vec{q}_N) \right\}$ $\quad$ space dim $= d$

(U: pot. energy.)

If we deal with identical particles, the many-particle wave funct° must be symmetric - For 2-particles $(\vec{k}_1, \vec{k}_2)$ & $(\vec{k}_2, \vec{k}_1)$ correspond to the same state! → Pay attention not to overcount!

Rule $\quad \phi(E) = \dfrac{1}{N!} \dfrac{1}{h^{dN}} \int d^d q_1 \cdots d^d q_N\, d^d \vec{p}_1 \cdots d^d p_N\, \Theta_H\left(E - H(\{\vec{q}_i, \vec{p}_i\})\right)$

$$\phi(E) = \frac{1}{N!} \int \frac{d^d q_1\, d^d p_1}{h^d} \int \frac{d^d q_2\, d^d p_2}{h^d} \cdots \int \frac{d^d q_N\, d^d p_N}{h^d}\ \Theta_H(\text{---})$$

Ex: N free identical particles in $d=3$

$$\phi(E) = \frac{1}{N!} \frac{V^N}{\Gamma\left(1+\frac{3N}{2}\right)} \left(\frac{mE}{2\pi\hbar^2}\right)^{\frac{3N}{2}} \Rightarrow \rho(E) \propto E^{\frac{3N}{2}-1}$$

with $N \sim 10^{23}$!

exp(exp) inverse of $\rho(E)$!