TD2 Phase Space & Ergodicity

2.4) Ergodicity for a sphere in a fluid

$$\begin{cases} \dot{x}(t) = v(t) \\ m\dot{v}(t) = -\gamma v(t) - k x(t) + F(t) \end{cases} \qquad \langle F(t) \rangle = 0$$

$T = 2\pi\sqrt{\frac{m}{k}}$ $\qquad \tau_{relax} = m/\gamma \ll T \rightarrow$ strong frict°

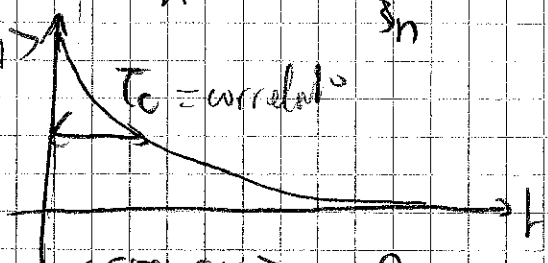
A) 1) $k = 0 \Rightarrow V_n = v(n\tau)$ with $\tau \ll \tau_{relax}$

$$\dot{v} = \frac{V_{n+1} - V_n}{\tau}$$

$$m\dot{v} = m\frac{(V_{n+1} - V_n)}{\tau} = -\gamma v_n + F(n\tau)$$

$$\Rightarrow V_{n+1} = V_n\underbrace{\left(1 - \frac{\gamma\tau}{m}\right)}_{\lambda} + \underbrace{\frac{\tau}{m}F_n}_{\xi_n} = \lambda V_n + \xi_n$$

2) Typically $\langle F(t) F(0) \rangle$



If $\underline{\tau \gg \tau_c}$ $\quad \langle F(\tau) F(0) \rangle \simeq 0 \rightarrow$ decorrelated

$\hookrightarrow \tau_{relax} \gg \tau \gg \tau_c$ hierarchy of time scale

In this limit $\langle F(n\tau) F(m\tau) \rangle \propto \delta_{n,m}$

3) $\langle F(n\tau)^2 \rangle = \frac{\sigma^2}{\tau} \rightarrow$ Gaussian distribut°

$$V_{n+1} = \lambda V_n + \xi_n = \dots = \lambda^{n+1} V_0 + \sum_{k=0}^{n} \lambda^k \xi_{n-k} \rightarrow \text{Eq (3)}$$

$$\text{var}(V_n) = \langle V_n^2 \rangle - \langle V_n \rangle^2 = \lambda^{2n}\,\text{var}(V_0) + 2\lambda^n \sum_k \lambda^k \underbrace{\langle V_n \xi_{n-k} \rangle}_{=0}$$

$$+ \left(\sum_{k,k'} \lambda^{k+k'} \underbrace{\langle \xi_{n-k}\, \xi_{n-k'} \rangle}_{= \delta_{k,k'}}\right)$$

$$\mathrm{var}(v_n) = \lambda^{2n}\,\mathrm{var}(v_0) + \sum_{l=0}^{n-1} \lambda^{2l}\, \left(\frac{\tau}{m}\right)^2 \frac{\sigma^2}{\tau}$$

$$\mathrm{var}(v_n) = \lambda^{2n}\,\mathrm{var}(v_0) + \frac{1-\lambda^{2n}}{1-\lambda^2}\,\frac{\sigma^2 \tau}{m^2}$$

$$\left.\begin{aligned}\lambda = 1-\frac{\gamma\tau}{m} = 1-\frac{\tau}{\tau_{relax}}\\ <1\end{aligned}\right\} \lambda^{2n} \longrightarrow 0 \Rightarrow \mathrm{var}(v_n) \to \frac{\sigma^2 \tau/m^2}{1-(1-\frac{\gamma\tau}{m})^2}$$

$$\Rightarrow \underline{\mathrm{var}(v_n) \approx \frac{\sigma^2}{2m\gamma}}$$ independ of initial cond.

$\to v_n$ becomes stationary and also Gaussian

$$P_t(v) \xrightarrow[t \gg t_n]{} P_{eq}(v) \text{ where } P_{eq}(v) = \frac{1}{\sqrt{2\pi}\,\sigma_v} e^{-\frac{v^2}{2\sigma_v^2}}$$

4) Fluctuation-dissipat°

$$\left.\begin{aligned}\langle E_c\rangle &= \tfrac{1}{2} k_B T\\ \tfrac{1}{2} m \langle v^2\rangle &\end{aligned}\right\} \langle v^2\rangle = \frac{k_B T}{m} = \frac{\sigma^2}{2m\gamma} \Rightarrow \boxed{\frac{\sigma^2}{2\gamma} = k_B T}$$

temperature

dissipat°

5) We have $\sigma_v^2 = \frac{k_B T}{m} = \frac{\sigma^2}{2m\gamma} \Rightarrow P_{eq} \propto e^{-\frac{v^2}{2\sigma_v^2}}$

$$= e^{-\frac{v^2 \gamma m}{\sigma^2}} = \underline{e^{-\frac{E_c}{k_B T}}}$$ Maxwell distribut° funct°

Posit° relaxion

overdamped $\underset{0}{m\ddot{x}} = -\gamma \dot{x} - kx + F(t) \Rightarrow \dot{x} = \frac{k}{\gamma} x + \frac{1}{\gamma} F(t)$

we solve $\dot{v} = -\frac{\gamma}{m} v + \frac{1}{m} F(t)$ (same type of eq. here!)

Therefore substitute $\gamma \to k$
$m \to \gamma$

$$\Rightarrow \underline{\mathrm{var}(x(t)) = \frac{\sigma^2}{2\gamma k}}$$

$$\sigma_x^2 = \mathrm{var}(x(t)) = \frac{k_B T}{k} \to P_{eq}(x) \propto e^{-\frac{E_p(x)}{k_B T}}$$

$$\rho_{eq}(x,p) \propto \rho_{eq}(x)\,\rho_{eq}(p) \propto e^{-(E_c+E_p)/k_B T}$$

2.5)

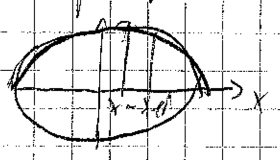
A) 1) $H(x,p) = \frac{p^2}{2m} + \frac{1}{2} m w^2 x^2$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \ ; \ \dot{p} = -\frac{\partial H}{\partial x} = -m w^2 x$$

$$\hookrightarrow \ddot{p} = -m w^2 \dot{x} = -\cancel{m} w^2 \frac{p}{m} = -w^2 p \Rightarrow \ddot{p} + w^2 p = 0 \Leftrightarrow \ddot{x} + w^2 x = 0$$

$$\Rightarrow x(t) = x_0 \cos(wt) \ ; \ p = -m x_0 w \sin(wt)$$

2)



$E = \underbrace{\frac{1}{2} m w^2 x^2 + \frac{1}{2m} p^2}_{\frac{1}{2} m w^2 x_0^2 \text{ conserved}}$ → ellipse

3)

$$\frac{dx}{dt} = -w x_0 \sin(wt) \Rightarrow dx = -w x_0 \sin(wt) dt = \frac{p}{m} dt$$

$$\Rightarrow dt = \frac{m}{p} dx = \frac{m}{\sqrt{2mE - m^2 w^2 x^2}} dx$$

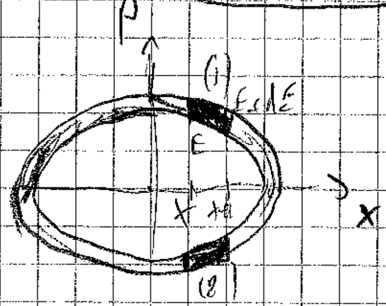
⚠ It takes a time $T = \frac{\pi}{w}$ for $\frac{1}{2}$ cycle → Be careful, pass twice

$$\Rightarrow \text{Therefore } \frac{dt}{T} = w(x) dx \Rightarrow w(x) dx = \underbrace{\frac{m w}{\pi \sqrt{2mE - m^2 w^2 x^2}}}_{w(x)} dx$$

$$\Rightarrow \boxed{w(x) = \frac{m w}{\pi \sqrt{2mE - m^2 w^2 x^2}}}$$

B) Slot 4

1)



2)

$P_x dx = \frac{2 \text{ [redacted]}}{}$

Area of a elliptical ring $A = \pi a b$ for an ellipse defined $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

we have $\frac{p^2}{2mE} + \frac{x^2}{2E/mw^2} = 1$

$$a = \sqrt{2mE} \ ; \ b = \sqrt{\frac{2E}{mw^2}}$$

$$A = \frac{2\pi E}{w} = E \times T \Rightarrow \boxed{\delta A = \delta E \times T}$$

$$P(x)\,dx = \frac{2\,(dx)\,dp}{T\,dE} = \frac{2\,dx\,dp}{T\,dE} = \frac{2dx\left(\frac{m}{p}\right)\cancel{dE}}{T\,\cancel{dE}}$$

$$P(x)\,dx = \frac{m\omega\,dx}{\pi\sqrt{2mE - m^2\omega^2x^2}} = w(x)\,dx$$

Same result!

Time average = Stat average over phase space!