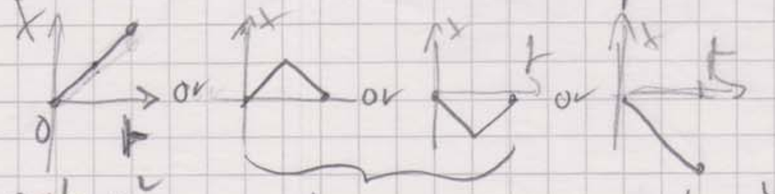
TD1: Random walk

Walker goes $\underline{\text{right}}$ with proba p see Fig 1.1
$\underline{\text{left}}$ $1-p = q$

1) $\Pi_M(n)$ = probability that the walker has made n steps to the right

Ex: $M=2$ 4 cases



proba p^2 proba $p \times q$ 2 independ   proba q^2

$$\Pi_M(n) = C_M^n p^n \underbrace{(1-p)}_{q}^{M-n} = \frac{M!}{n!(M-n)!} p^n q^{M-n}$$

$$\sum_{n=0}^{M} \Pi_M(n) = \sum_n C_M^n p^n q^{M-n} = (p+q)^M = 1 \quad \underline{\text{ok}}$$

2) $$\langle n^q \rangle = \sum_{n=0}^{M} n^q \, \Pi_M(n) = ??? \to \text{ok for } \underline{q=1} \to \langle n \rangle = pM$$

3) Let us introduce $G_M(s) \equiv \langle s^n \rangle = \sum_{n=0}^{M} s^n C_M^n p^n q^{M-n}$

a) $G_M'(s) = \langle n s^{n-1} \rangle \Rightarrow \langle n \rangle = G_M'(s)\big|_{s=1}$

$$\langle n(n-1) \rangle = \partial_s^2 G_M(s)$$
$$= \langle n^2 \rangle - \langle n \rangle \to \text{we have access to } \langle n^2 \rangle$$

etc... $\Rightarrow$ all moments

b) $$G_M(s) = \sum_{n=0}^{M} C_M^n (ps)^n q^{M-n} = (ps+q)^M \text{ binom}$$
$$= (1 + p(s-1))^M$$

$$\langle n \rangle = G_M'(s=1) = M(1+s(p-1))^{M-1} \times p\big|_{s=1} = Mp$$

$$\langle n^2 \rangle - \langle n \rangle = M(M-1)(1+s(p-1))^{M-2} \times p^2\big|_{s=1} = M(M-1)p^2$$

$$\langle n^2 \rangle = M\left[(M-1)p^2 + p\right]$$

$$\mathrm{Var}(n) \equiv \langle n^2\rangle - \langle n\rangle^2 = \langle n^2\rangle - \langle n\rangle + \langle n\rangle - \langle n\rangle^2$$
$$= M(M-1)p^2 + Mp - M^2p^2$$
$$= Mp(1-p) = Mpq$$
$$\sigma = \sqrt{\mathrm{var}(n)} = \sqrt{M}\sqrt{pq}$$
while $\langle n\rangle \sim Mp \to \dfrac{\sigma}{\langle n\rangle} \sim M^{-1/2}$

c) $W_M(\beta) = \ln G_M(e^{-\beta})$
$$= \ln \langle e^{-\beta n}\rangle$$
$$= \ln \langle 1 - \beta n + \frac{\beta^2}{2} n^2 + \dots \rangle$$
$$= \ln \left(1 - \beta\langle n\rangle + \frac{\beta^2}{2}\langle n^2\rangle + \dots\right)$$
$$= -\beta\langle n\rangle + \frac{\beta^2}{2}\langle n^2\rangle - \frac{\beta^2}{2}\langle n\rangle^2 + \dots$$
$$= -\beta\langle n\rangle + \frac{\beta^2}{2}(\mathrm{var}(n)) + \dots$$

Application $\ln\langle e^{-\beta n}\rangle = \ln \sum_{n=0}^{M} C_M^n e^{-\beta n} p^n q^{M-n} = \ln\left(pe^{-\beta}+q\right)^M$
$$= M\ln\left(pe^{-\beta} + 1 - p\right)$$
$$= M\ln\left(1 - p(1-e^{-\beta})\right)$$
$$= M\ln\left(1 - p\left(\beta - \frac{\beta^2}{2} + \dots\right)\right)$$
$$= M\left[-\beta p + \frac{\beta^2}{2}(p - p^2) + O(\beta^3)\right]$$
$$= -\beta \underbrace{(Mp)}_{\langle n\rangle} + \frac{\beta^2}{2}\underbrace{Mp(1-p)}_{\mathrm{var}(n)} + O(\beta^3)$$

we directly read the mean and the variance!

d) Limit $M \to \infty$
$$\ln \Pi_M(n) = \ln(C_M^n) + n\ln p + (M-n)\ln q$$
$$= \ln M! - \ln n! - \ln(M-n)! + n\ln p + (M-n)\ln q$$
stirling $\simeq M\ln M - n\ln n - (M-n)\ln(M-n) + n\ln p + (M-n)\ln q$

Find the maximum of $\ln \Pi_M(n)$

$$\frac{d}{dn}\ln \Pi_M(n) = -\ln n + \ln(M-n) + \ln p - \ln q$$

$$= 0 \quad \text{for} \quad \ln\left(\frac{M-n^*}{n^*}\right) = \ln\left(\frac{1-p}{p}\right) \Rightarrow n^* = pM$$

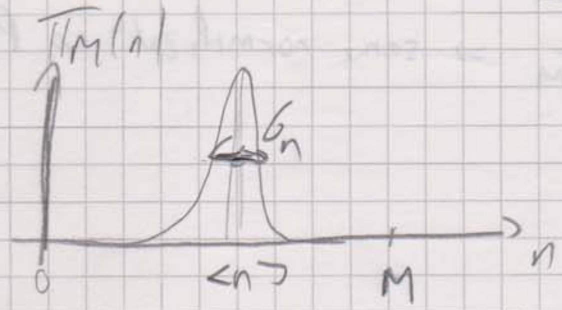
$$\left.\frac{d^2}{dn^2}\ln \Pi_M(n)\right|_{n^*} = \left. -\frac{1}{n} - \frac{1}{M-n}\right|_{n^*} = \left. -\frac{M}{n(M-n)}\right|_{n^*} = \frac{-M}{pM^2(1-p)} = -\frac{1}{Mpq}$$

$$\Rightarrow \ln \Pi_M(n) \simeq \ln \Pi_M(n^*) + \frac{1}{2}(n-n^*)^2\left(\frac{d^2 \ln \Pi_M}{dn^2}\right)_{n^*} + \ldots$$

$$\simeq \ln \Pi_M(n^*) - \frac{(n-n^*)^2}{2Mpq} + \ldots$$

$$\Pi_M(n) \simeq \Pi_M(n^*)\, \underbrace{\exp\left(-\frac{(n-n^*)^2}{2Mpq}\right)}$$

Gaussian distribut° of variance Mpq



valid when we are far away from the borders ⇒ narrow distribut°

valid $\Rightarrow$ $\sigma_n \ll \langle n \rangle$ and $\sigma_n \ll M - \langle n \rangle$

$$\sqrt{Mpq} \ll Mp$$

$$\Leftrightarrow \sqrt{\frac{1-p}{p}} \ll \sqrt{M} \Rightarrow \frac{1}{p} \ll M$$

and $\sigma_n \ll M - \langle n \rangle \rightarrow \frac{1}{q} \ll M$

Therefore, we should have $p \gg \frac{1}{M}$ and $q \gg \frac{1}{M}$

$\Rightarrow$ avoid p very small or q very small

Part B. Walker 1) $x = na + (M-n)(-a) = (2n-M)a$

$$\langle x\rangle = (2\langle n\rangle - M)a = (2p-1)Ma$$

If $p > \frac{1}{2}$, $\langle x\rangle > 0$ as expected

$$\mathrm{var}\left(\frac{x}{a}\right) = \mathrm{var}(2n-M) = \mathrm{var}(2n) = 4\,\mathrm{var}\,n$$
$$= 4Mpq \Rightarrow \boxed{\mathrm{var}(x) = 4a^2Mpq}$$

Rq: $\mathrm{var}(X) = \overline{X^2} - \overline{X}^2 = \overline{(X-\overline{X})^2}$

Therefor $\mathrm{var}(X - X_0) = \overline{(X - X_0 - \overline{X - X_0})^2} = \overline{(X-\overline{X})^2} = \mathrm{var}X$
with $X_0 = \text{const}$

2) $$V \equiv \lim \frac{\langle x\rangle_t}{t} = \lim_{\substack{M\to\infty \\ t = M\tau}} \frac{(2p-1)Ma}{M\tau} = \boxed{\frac{(2p-1)a}{\tau}}$$

If $p = \frac{1}{2}$, $V = 0$ as expected

3) $$D = \lim_{M\to\infty} \frac{\mathrm{var}\,x}{2M\tau} = \frac{4a^2Mpq}{2M\tau} = \frac{2a^2pq}{\tau}$$

4) We have seen $\Pi_M(n) \propto \exp\left(-\frac{(n-\langle n\rangle)^2}{2Mpq}\right)$

and $n - \langle n\rangle = \frac{x - \langle x\rangle}{2a}$

$$\hookrightarrow P_t(x) \propto \exp\left(-\frac{(x-\langle x\rangle)^2}{4a^2 \cdot 2Mpq}\right) = \exp\left(-\frac{(x-\langle x\rangle)^2}{8a^2\frac{t}{M}pq}\right)$$

Using 3) $$\boxed{P_t(x) = \mathcal{N} \exp\left(-\frac{(x-\langle x\rangle)^2}{4Dt}\right)}$$

with $\mathcal{N}$ normalizat$^{\circ}$ $\Rightarrow \mathcal{N} = \frac{1}{\sqrt{4\pi Dt}}$ (Gaussian)

Let us compare $\Pi_M(n)$ and $P_t(x)$ for $p = 1/2$

We have established $\Pi_M(n) \simeq \Pi_M(n^*) \exp\left(-\frac{(n-n^*)^2}{2Mpq}\right)$

$$\Pi_M(n^*) = \frac{M!}{n^*!\,(M-n^*)!}\left(\frac{1}{2}\right)^{n^*}\left(\frac{1}{2}\right)^{M-n^*}$$

Therefore $\Pi_M(n^*) = \frac{M!}{n^*!\,(M-n^*)!}\left(\frac{1}{2}\right)^M = \frac{M!}{\left(\frac{M}{2}!\right)^2}\frac{1}{2^M}$

Using the Stirling formula $\ln \Pi_M(n^*) \simeq M\ln M - M\ln\frac{M}{2}$

$$- M\ln 2 + \ln\left(\frac{\sqrt{2\pi M}}{\pi M}\right)$$

$$\Rightarrow \ln \Pi_M(n^*) \simeq \ln\sqrt{\frac{2}{\pi M}} \rightarrow \Pi_M(n^*) \simeq \left(\frac{2}{\pi M}\right)^{1/2}$$

Therefore $\Pi_M(n^*) \simeq \sqrt{\frac{2}{\pi M}}\exp\left(-\frac{(x-\overset{0}{\langle x\rangle})^2}{2Ma^2}\right)$

We have $x = (2n-M)a = 2na - Ma = 2a(n-n^*)$

$$\frac{x}{2a} = n - n^*$$

$$\Rightarrow \frac{1}{2a}\Pi_M(n^*) = \sqrt{\frac{1}{2\pi Ma^2}}\exp\left(-\frac{x^2}{2Ma^2}\right) = P_t(x^*)$$

We established that $P_{t=M\tau}(x) = \frac{\Pi_M}{2a}(n)$

The factor $2a$ comes from the dimension of x

c) Steps as $\subseteq$ random variable (we can skip this part) too difficult

1) hop h $\rightarrow x$ $\quad x \to x+h$ with h distributed with $p(h)$

$P_{t+\tau}(x)$ = Proba of being at posit° $x-h$ at time t / $P_t(x-h)$
and of making a hopping of length h / $p(h)$ / $\forall h$

$$\underline{P_{t+\tau}(x) = \int dh \, P_t(x-h)\, p(h)} \Big\| = P_t * p(x)$$

2) $p_\sigma(h) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{h^2}{2\sigma^2}\right)$

$$P_{t+\tau}(x) = \int dh \; P_t(x-h)\, p_\sigma(h) = P_t * p_\sigma(x)$$

Suppose at $t=0$ · $P_0(x) = \delta(x)$ → origin

$$P_\tau(x) = \int dh \; P_0(x-h)\, p_\sigma(h) = \int dh\, \delta(x-h)\, p_\sigma(h) = p_\sigma(x)$$

$$P_{2\tau}(x) = \int dh \; p_\sigma(x-h)\, p_\sigma(h) = p_\sigma * p_\sigma(x)$$

$$\boxed{P_{M\tau}(x) = \underbrace{p_\sigma * \cdots * p_\sigma}_{M\text{ times}}(x) = p_{\sqrt{M}\sigma}(x) = \frac{1}{\sqrt{2\pi M\sigma^2}}\, e^{-\frac{x^2}{2M\sigma^2}}}$$

Detour: Convolut° of Gaussians

Fourier transform: $\tilde{f}(q) = \int dx \, e^{-iqx} f(x)$

$$f(x) = \int \frac{dq}{2\pi}\, e^{iqx}\, \tilde{f}(q)$$

. Fourier transform of a Gaussian

$$\tilde{g_\sigma}(q) = \int dx \; e^{-iqx}\, g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int dx \, e^{-iqx}\, e^{-\frac{x^2}{2\sigma^2}}$$

$$= e^{-\sigma^2 q^2/2}$$

. Fourier transform of a convolut°
= product of Fourier transforms.) show it

$$\tilde{P}_{M\tau}(q) = (\tilde{p}_\sigma(q))^M = e^{-M\sigma^2 q^2/2} = e^{-(\sqrt{M}\sigma)^2 q^2/2}$$

Inverse Fourier transform → Give a Gaussian with variance $\tilde{\sigma}_M = \sqrt{M}\sigma$ ||

3) Method 2 $\langle h\rangle < \infty$ $\langle h^2\rangle < \infty$

We consider $\tau \to 0$ and small steps (width of $p(h) \to 0$)

a) $p(h) \propto e^{-\frac{h^2}{2\sigma^2}}$ → Gaussian

$p(h) = \frac{1}{2a} e^{-\frac{|h|}{a}}$ $a > 0$ Exp → need to make it normalisable $\underline{\forall h}$

b) $P_{t+\tau}(x) = \int dh\, p(h)\, P_t(x-h)$

$$P_{t+\tau}(x) \simeq P_t(x) + \tau\, \partial_t P_t(x)$$

$$= \int dh\, p(h)\, P_t(x) - \int dh\, h\, p(h)\, \partial_x P_t(x) + \frac{1}{2}\int dh\, h^2 p(h)\, \partial_x^2 P_t$$

$$= P_t(x) - \langle h\rangle \partial_x P_t(x) + \frac{1}{2}\langle h^2\rangle \partial_x^2 P_t(x) + \dots$$

c) ⇒ $\tau\, \partial_t P_t(x) \overset{+O(\tau^2)}{\simeq} -\langle h\rangle \partial_x P_t(x) + \frac{1}{2}\langle h^2\rangle \partial_x^2 P_t(x) + \dots$

Suppose $\tau \propto \varepsilon$; $\langle h\rangle \propto \varepsilon$; $\langle h^2\rangle \propto \varepsilon$

$$\partial_t P_t(x) = -\frac{\langle h\rangle}{\tau} \partial_x P_t(x) + \frac{1}{2}\underbrace{\frac{\langle h^2\rangle}{\tau}}_{\sim \frac{1}{2\tau}(\langle h^2\rangle - \langle h\rangle^2)} \partial_x^2 P_t(x)$$

$\sim \frac{1}{2\tau}\left(\langle h^2\rangle - \underbrace{\langle h\rangle^2}_{O(\varepsilon^2)}\right)\partial_x^2 P_t(x)$

$$\Rightarrow \boxed{\partial_t P_t(x) = -V\partial_x P_t(x) + D\partial_x^2 P_t(x)}$$

Diffusion eq.

d) Fourier transform $\tilde{P}_t(q) = \int e^{-iqx} dx\, P_t(x) \Leftrightarrow P_t(x) = \int e^{iqx} \frac{dq}{2\pi} \tilde{P}_t(q)$

$$\int dx\, e^{-iqx}\, \partial_t P_t(x) = -V\int dx\, e^{-iqx}\, \partial_x P_t(x) + D\int dx\, e^{-iqx}\, \partial_x^2 P_t(x)$$

$$\partial_t \tilde{P}_t(k) = -ikV\, \tilde{P}_t(k) - Dk^2\, \tilde{P}_t(k)$$

$$= -(ikV + Dk^2)\, \tilde{P}_t(k)$$

$$\tilde{P}_t(k) = \exp\left(-(ikV + Dk^2)t\right)$$

$$P_t(x) = \int \frac{dk}{2\pi} e^{ikx} e^{-ikVt - Dk^2 t} = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-Vt)^2}{4Dt}}$$

→ $\boxed{\sigma = 2Dt}$

4) Universality → In the Σ limit, all models lead to a Gaussian distribut° ⇒ This is a consequence of the central limit thm
Actually, our demonstrat° in 3) rely on a small nb of assumptions → almost a proof of the thm

D) d-dimensional case

1) $\delta\vec{x} = \sum h_i \vec{e}_i$ → h_i independent random variables

we are interested in $P_t(\vec{x})$

After M steps, $\vec{x} = \sum_{n=1}^{M} h_{1,n}\vec{e}_1 + \sum_{n=1}^{M} h_{2,n}\vec{e}_2 + \dots \sum_{n=1}^{M} h_{d,n}\vec{e}_d$

$\hookrightarrow P_t(\vec{x}) = P_t(x_1)\, P_t(x_2) \dots P_t(x_d)$ where $P_t(x_i) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x_i^2}{4Dt}}$

independent variables!

2) $\langle \vec{x}^2 \rangle = \sum_{i=1}^{d} \langle x_i^2 \rangle + 2 \sum_{i\neq j} \underbrace{\langle x_i x_j \rangle}_{0} = \langle x_i \rangle \langle x_j \rangle$

$\langle x_i^2 \rangle = \int dt\, x_i^2\, P_t(x_i) = \sigma^2 = 2Dt$

$\hookrightarrow \boxed{\langle \vec{x}^2 \rangle = 2Ddt}$

3) $\vec{x} \in \mathbb{R}^2 \to P_t(\vec{x})$; $Q_t(r = \sqrt{x^2+y^2})$

$P_t(\vec{r}) = \frac{1}{4\pi Dt} \exp\left(-\frac{x^2+y^2}{4Dt}\right)$

Conservat° of probability

$$\boxed{\underbrace{Q_t(r)\,dr}_{\text{Proba}\{\sqrt{x^2+y^2} \in [r, r+dr]\}} = r\,dr \int_0^{2\pi} d\theta\, P_t(x,y)}$$

$Q_t(r)\,dr = r\,dr\, \frac{2\pi}{2\cdot 4\pi Dt} e^{-\frac{r^2}{4Dt}}$

$$\boxed{Q_t(r) = \frac{r}{2Dt} e^{-r^2/4Dt}} \to \boxed{\int_0^{+\infty} Q_t(r)\,dr = 1}$$

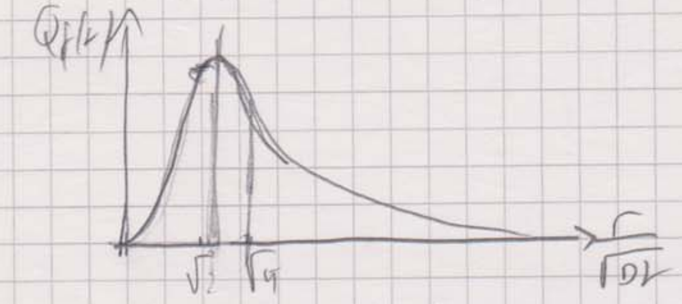
$$Q_t'(r) = \frac{1}{2Dt}\left(1 - \frac{r^2}{2Dt}\right) e^{-r^2/4Dt} = 0$$

$$r_{typ} \sim \sqrt{2Dt}$$

$$\langle r \rangle = \int_0^{+\infty} dr \; r \, Q_t(r) = \int_0^{\infty} dr \, r^2 e^{-\frac{r^2}{4Dt}} = \sqrt{\pi D t}$$

similarly $\langle r^2 \rangle = 4Dt$

$$\mathrm{var}(r) = (4-\pi) Dt \rightarrow \sigma = \underbrace{\sqrt{4-\pi}}_{0.92}\sqrt{Dt} \sim \sqrt{Dt}$$



4) $v = 500\,ms^{-1}$; one chock every $\tau = 2ns$

$\hookrightarrow$ 1 chock every $\ell = 1\mu m$ ||

In 1s : . Ballistic motion $\rightarrow$ $\Delta x_{bal} = vt = \pm 500m$ ||

. Diffusive motion $\rightarrow$ $M = \frac{t}{\tau} = 5\,10^8$ chocks

$$\sigma \sim \Delta x \sim \sqrt{Dt} \sim \sqrt{\frac{\ell^2}{\tau} t} = \ell\sqrt{\frac{t}{\tau}} = \ell\sqrt{M}$$

$$\Delta x|_{diff} = \sqrt{M}\,\ell = 5\,10^4 \cdot 10^{-6} = 5\,10^{-2}\,m \sim 5cm \ll \Delta x|_{bal}$$

here we used $D \sim \frac{a^2}{\tau}$

with $a \equiv \ell$