Exercice B

1) $$\hat{H}_1 = -\vec{\hat{\mu}}\cdot\vec{B} = -\gamma B_x(t)\vec{S}_x - \gamma B_y(t)\vec{S}_y - \gamma B_z\vec{S}_z$$

$$= \underbrace{-\gamma B_1}_{\omega_1}\frac{\hbar}{2}\left(\cos(\omega t)\hat{\sigma}_x + \sin(\omega t)\hat{\sigma}_y\right) + \underbrace{-\gamma B_0}_{\omega_0}\frac{\hbar}{2}\hat{\sigma}_z$$

Rappel: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

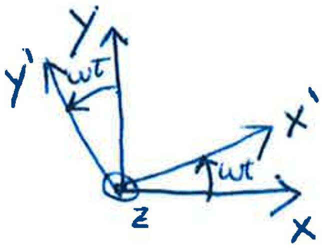
$$\Rightarrow \hat{H}_1 = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix} \quad ; \quad \omega_0, \omega_1, \omega \geq 0.$$

2) $$V(t) = e^{+\frac{i}{\hbar}S^z\omega t} = \begin{pmatrix} e^{+i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}$$

Example of the action of $V(t)$

$$\underbrace{\frac{\hbar}{2}\hat{\sigma}'_x}_{} = V(t)\underbrace{\frac{\hbar}{2}\hat{\sigma}_x}_{} V(t)^\dagger$$

$\hat{S}_x$ in the laboratory frame of reference/representation



The operator $\hat{S}_x$ described in the rotating frame of reference.

Be careful! $\frac{\hbar}{2}\hat{\sigma}'_x \neq \frac{\hbar}{2}\hat{\sigma}_{x'}$

In the rotating frame of reference
$\hat{\sigma}_{x'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Explicit calculation:

$$\hat{\sigma}_x' = \hat{V}(t)\,\hat{\sigma}_x\,\hat{V}(t)^\dagger = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}$$

Example $\omega t = \frac{\pi}{2} \longrightarrow \hat{\sigma}_x' = -\hat{\sigma}_{y'}$ CORRECT!

$\omega t = \pi \longrightarrow \hat{\sigma}_x' = -\hat{\sigma}_{x'}$ CORRECT!

$\omega t = \frac{3\pi}{2} \longrightarrow \hat{\sigma}_x' = \hat{\sigma}_{y'}$ CORRECT!

3) $$\hat{H}_2 = \hat{V}(t)\,\hat{H}_1(t)\,\hat{V}(t)^\dagger + i\hbar \frac{d\hat{V}(t)}{dt}\hat{V}(t)^\dagger$$

- $$\frac{d\hat{V}(t)}{dt} = \frac{i}{\hbar}\hat{S}^z\omega\,\hat{V}(t) \Rightarrow i\hbar\frac{d\hat{V}(t)}{dt}\hat{V}(t)^\dagger = -\omega\hat{S}_z = -\frac{\hbar\omega}{2}\hat{\sigma}_z$$

- $$\hat{V}(t)\,\hat{H}_1(t)\,\hat{V}(t)^\dagger = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix}\begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix} =$$

$$= \frac{\hbar}{2}\begin{pmatrix} \omega_0 e^{i\omega t/2} & \omega_1 e^{-i\omega t/2} \\ \omega_1 e^{i\omega t/2} & -\omega_0 e^{-i\omega t/2} \end{pmatrix}\begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix} =$$

$$= \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 \\ \omega_1 & -\omega_0 \end{pmatrix}$$

$$\hat{H}_2 = \frac{\hbar}{2}\begin{pmatrix} \omega_0 - \omega & \omega_1 \\ \omega_1 & \omega - \omega_0 \end{pmatrix}$$

4) $|\Psi_1(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow |\Psi_2(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\uparrow$ Laboratory. Representation R_1

$\uparrow$ Rotating frame of reference. Representation R_2.

$$|\Psi_2(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_2 t}\,|\Psi_2(0)\rangle$$

Si $\omega = \omega_0$: $\quad H_2 = \frac{\hbar\omega_1}{2}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$ $\rightarrow$ $\hat{\sigma}_{x'}$ in the representation R_2

$$e^{-i/\hbar\,\hat{H}_2 t} = \sum_{k=0}^{+\infty} \frac{\left(-i/\hbar\; t\; \frac{\hbar\omega_1}{2}\right)^k}{k!}\left[\hat{\sigma}_{x'}\right]^k \qquad \text{Remarq: } \sigma_{x'}^2 = \mathbb{1}$$

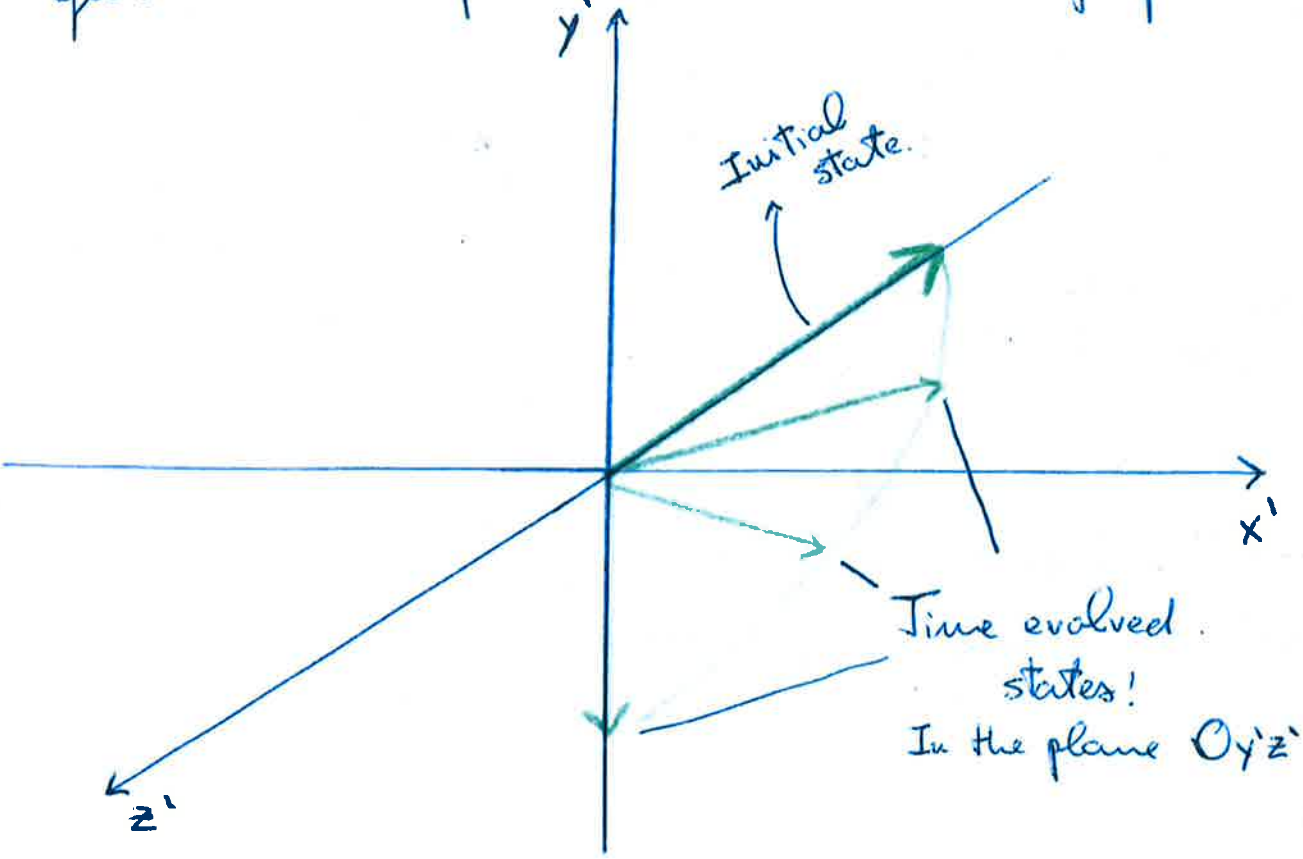
$$= \hat{\mathbb{1}}\cdot\sum_{k\,\text{even}} \frac{(-1)^{k/2}\left(\frac{\omega_1 t}{2}\right)^k}{k!} - i\hat{\sigma}_{x'}\sum_{k\,\text{odd}} \frac{(-1)^{\frac{k-1}{2}}\left(\frac{\omega_1 t}{2}\right)^k}{k!} =$$

$$= \hat{\mathbb{1}}\cdot\cos\left(\frac{\omega_1 t}{2}\right) - i\,\hat{\sigma}_{x'}\sin\left(\frac{\omega_1 t}{2}\right)$$

$$= \begin{pmatrix} \cos\left(\frac{\omega_1 t}{2}\right) & -i\sin\left(\frac{\omega_1 t}{2}\right) \\ -i\sin\left(\frac{\omega_1 t}{2}\right) & \cos\left(\frac{\omega_1 t}{2}\right) \end{pmatrix}$$

$$|\Psi_2(t)\rangle = \begin{pmatrix} -i\sin\left(\frac{\omega_1 t}{2}\right) \\ \cos\left(\frac{\omega_1 t}{2}\right) \end{pmatrix}$$

Geometric interpretation in the rotating frame. 5



The time evolution is a rotation around the axis x' with angular velocity $-\omega_1$ (check for $\omega_1 t = \frac{\pi}{2}$, eigenstate of $\frac{\hbar}{2}\hat{\sigma}_{y'}$, $-\frac{\hbar}{2}$)

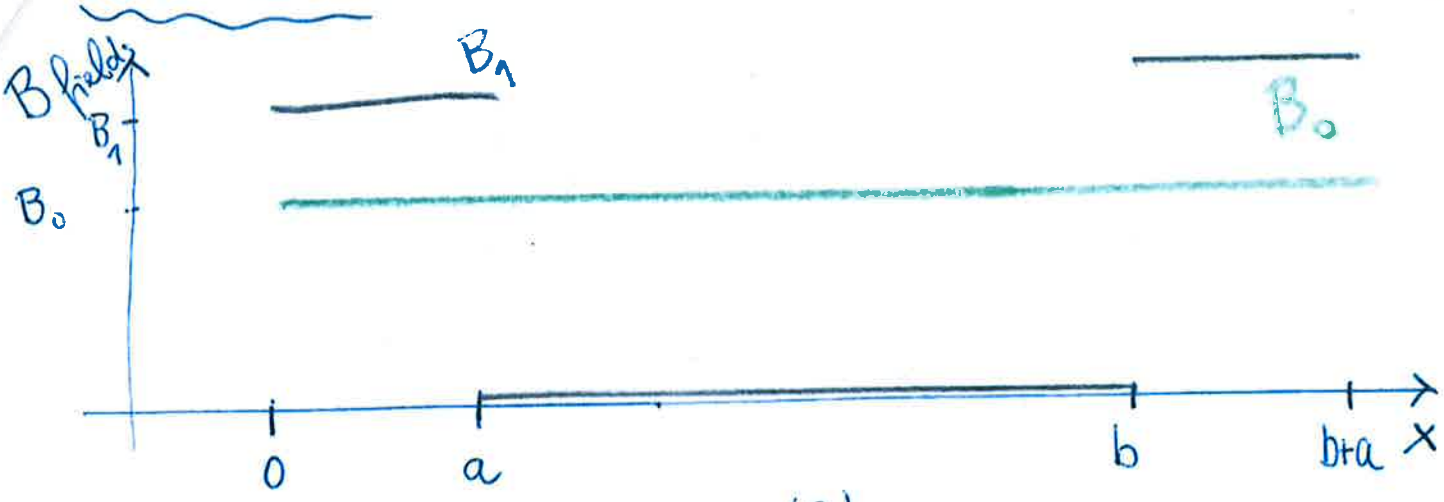
Be CAREFUL: The axis x' is rotating with angular velocity ω around the axis $z \equiv z'$

5) $$|\psi_1(t)\rangle = \hat{V}^+(t)\,|\psi_2(t)\rangle =$$

$$= \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix} \begin{pmatrix} -i \sin\frac{\omega_1 t}{2} \\ \cos\frac{\omega_1 t}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} -i\, e^{-i\omega t/2} \sin\frac{\omega_1 t}{2} \\ e^{i\omega t/2} \cos\frac{\omega_1 t}{2} \end{pmatrix}$$

Exercice C



$t=0, \quad x=0 \qquad |\Psi_1(t=0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ {Laboratory Representation}

1) $t = \frac{a}{v} \Rightarrow x = a$

We have to take into account the presence of the oscillating magnetic field. We use the results of exercice B and introduce the same rotating reference frame and the related representation R_2.

$$|\Psi_2(t=0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{\{Rotating representation\}}$$

$$|\Psi_2(t=\tfrac{a}{v})\rangle = \begin{pmatrix} i\sin\left(\frac{\omega_1 a}{2v}\right) \\ \cos\left(\frac{\omega_1 a}{2v}\right) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Data of the exercice

$$\frac{\omega_1 a}{2v} = \frac{\pi}{4}$$

$$|\Psi_1(t=\tfrac{a}{v})\rangle = V(t)^{\dagger} |\Psi_2(t=\tfrac{a}{v})\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i\, e^{-i\omega a/2v} \\ e^{i\omega a/2v} \end{pmatrix}$$

2) In both reference frames, the time evolution between a and b is simple.

Rotating reference: $H_2 = 0$ $\left(\begin{matrix}\text{we're taking}\\ \omega = \omega_0\end{matrix}\right)$

$$|\Psi_2(t=\tfrac{b}{v})\rangle = |\Psi_2(t=\tfrac{a}{v})\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

Laboratory $H_1 = \frac{\hbar}{2}\begin{pmatrix}\omega_0 & 0 \\ 0 & -\omega_0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}\omega & 0 \\ 0 & -\omega\end{pmatrix}$

$$|\Psi_1(t=\tfrac{b}{v})\rangle = e^{-i/\hbar \, \hbar/2 \, \omega \frac{(b-a)}{v} \hat{\sigma}_z} |\Psi_1(t=\tfrac{a}{v})\rangle =$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} i \; e^{-i\omega b/(2v)} \\ e^{i\omega b/(2v)} \end{pmatrix} = V(t)^{\dagger}|\Psi_2(t=\tfrac{b}{v})\rangle$$

3) Rotating reference: We use previous results

$$|\Psi_2(t=\tfrac{b+a}{v})\rangle = e^{-i/\hbar \, \hat{H}_2 \, a/v}|\Psi_2(t=b/v)\rangle =$$

$$= \begin{pmatrix}\cos\left(\frac{\omega_1 a}{2v}\right) & i\sin\left(\frac{\omega_1 a}{2v}\right) \\ i\sin\left(\frac{\omega_1 a}{2v}\right) & \cos\left(\frac{\omega_1 a}{2v}\right)\end{pmatrix}\begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix} =$$

$$\frac{\omega_1 a}{2v} = \frac{\pi}{4} \quad \longrightarrow \quad = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix}$$

$= $ eigenstate of $\hat{S}_z = \hat{S}_{z'}$, $+\frac{\hbar}{2}$

Laboratory

$$|\Psi_2(t=\tfrac{b+a}{v})\rangle = V_2(t=\tfrac{b+a}{v})^\dagger |\Psi_1(t=\tfrac{b+a}{v})\rangle = i e^{-i\omega(b+a)/2v} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Probability of detecting $|\uparrow_z\rangle = 1$

4) $|\omega-\omega_0| \ll 1$ but $\omega \neq \omega_0$

- The calculation of the ~~passage~~ time-evolution of the neutrons in the first region $x \in [0,a]$ goes as before because:

$$H_2 = \frac{\hbar}{2}\begin{pmatrix} \omega_0-\omega & \omega_1 \\ \omega_1 & \omega-\omega_0 \end{pmatrix} \approx \frac{\hbar}{2}\begin{pmatrix} 0 & \omega_1 \\ \omega_1 & 0 \end{pmatrix}$$

Thus $|\Psi_2(t=\tfrac{a}{v})\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$

- The calculation of the time-evolution of the neutrons in the second region $x \in [a,b]$ changes:

$$H_2 = \frac{\hbar}{2}\begin{pmatrix} \omega_0-\omega & 0 \\ 0 & \omega-\omega_0 \end{pmatrix}$$

$$|\Psi_2(t=b/v)\rangle = e^{-i/\hbar \, H_2 \frac{b-a}{v}} \begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i\, e^{-i(\omega_0-\omega)\frac{b-a}{2v}} \\ e^{i(\omega_0-\omega)\frac{b-a}{2v}} \end{pmatrix}$$

- The calculation of the time-evolution in the region $x \in [b, b+a]$ is similar to before:

$$|\psi_2(t = \tfrac{b+a}{v})\rangle = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \overbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} i\, e^{-i(\omega_0-\omega)\frac{b-a}{2v}} \\ e^{i(\omega_0-\omega)\frac{b-a}{2v}} \end{pmatrix}} =$$

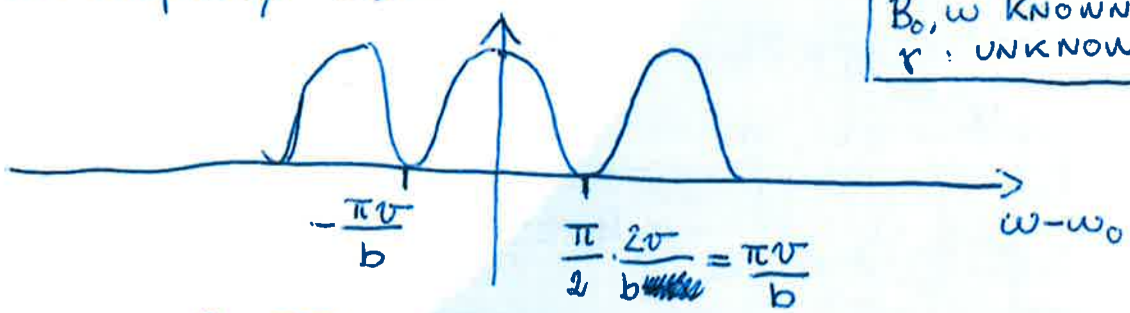
$$= \frac{1}{2} \begin{pmatrix} i e^{-ix} + i e^{ix} \\ -e^{-ix} + e^{ix} \end{pmatrix} = \begin{pmatrix} i \cos x \\ i \sin x \end{pmatrix}$$

$$x = \frac{(\omega_0-\omega)(b-a)}{2v}$$

$\Longrightarrow$ Probability of measuring $|\uparrow_z\rangle = \cos^2\left(\frac{(\omega_0-\omega)(b-a)}{2v}\right)$

- Oscillatory dependence on $\omega - \omega_0$
- Recall that $\omega_0 = -\gamma B_0$, related to the intensity of B_z. ω is the frequency of the rotation in the xy plane.
- For simplicity $b \gg a$

Typically:
B_0, ω KNOWN
γ : UNKNOWN



Largeur: $2 \times \frac{\pi v}{b}$

We can interpret the width of the central peak as an fundamental incertainty in the measurement of ω_0

Quantum mechanics: A FREQUENCY (energy) measurement is more precise if it takes a lot of time, in this case the measurement time is b/v. OK!

6) $\Delta\gamma$: incertitude for the measurement of γ.

$\Delta\omega$: incertitude " " " of ω

$$\Delta\gamma = \frac{\Delta\omega}{B_0} \cong \frac{2\pi v}{B_0 b}$$

$$b = \frac{2\pi v}{B_0 \Delta\gamma}$$

Data: $\Delta\gamma = 8{,}8 \cdot 10^{-7} \frac{q}{M_p}$

$v = h/\lambda_{dB} M_p \rightarrow \lambda_{dB} = 3{,}1 \cdot 10^{-9}\, m$

$B_0 = 1\, T$

$$b = \frac{2\pi\, 6{,}62 \cdot 10^{-34}}{3{,}1 \cdot 10^{-9} \cdot 2 \cdot 8{,}8 \cdot 10^{-7} \cdot 1{,}6 \cdot 10^{-19}} \frac{m^2\, Kg}{s} \cdot \frac{1}{m} \cdot \frac{C \cdot s}{Kg} \cdot \frac{1}{C} =$$

$$= \frac{2\pi \cdot 6{,}62}{3{,}1 \cdot 8{,}8 \cdot 1{,}6 \cdot 2} \cdot 10 \cdot m = 4{,}7\, m$$