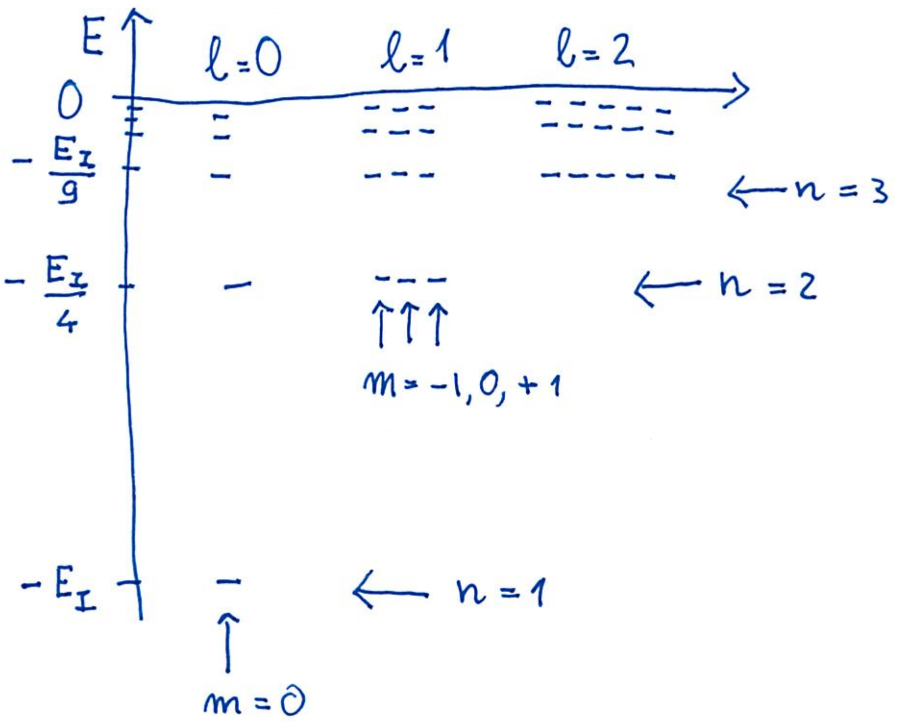
Exam _ May 2024.

Exercise 3.

1) $H_0 = -\frac{p^2}{2m_e} - \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$.

$E_n = -\frac{E_I}{n^2}$ with $n \in \mathbb{N}/\{0\}$. $E_I \simeq 13.6\,eV$

$g_n = n^2$




$|n, l, m\rangle$

or

$\psi_{n,l,m}(\vec{r})$

2) n determines the energy: principal quantum number

$$H\,|n, l, m\rangle = E_n\,|n, l, m\rangle$$

l ~~d~~ is associated to $L^2$

$$L^2\,|n, l, m\rangle = \hbar^2 l(l+1)\,|n, l, m\rangle$$

m is associated to $L_z$

$$L_z\,|n, l, m\rangle = \hbar m\,|n, l, m\rangle$$

3) $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$$xy = r^2\sin^2\theta\cos\phi\sin\phi = \frac{1}{2}r^2\sin^2\theta\sin 2\phi =$$

$$= -\frac{i}{4}r^2\sin^2\theta\left(e^{i2\phi} - e^{-i2\phi}\right)$$

4) From the writing above we obtain

$$\langle n', l', m' | V_1(\vec{r}) | n, l, m\rangle = -\frac{i}{4}\int_0^{+\infty} dr\, r^2 \cdot r^2 R^*_{n'l'}(r) R_{nl}(r) \cdot$$

$$\cdot \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \sin^2\theta \left(e^{i2\phi} - e^{-i2\phi}\right) Y^*_{l'm'}(\theta\phi) Y_{lm}(\theta\phi)$$

We recall that $Y_{lm} \propto e^{im\phi}$ therefore:

- When $\Delta m \neq 2, -2$ the matrix element is for sure zero.

5) When $\Delta m = 2, -2$ the matrix element could be different from zero.

6) The lowest energy level of the atom is not degenerate and is characterised by:

$$|n=1, l=0, m=0\rangle$$

At first order in $v_0$, the new eigenvalue reads:

$$E_{\text{ground state}} = -E_I + \langle n=1, l=0, m=0 | V | n=1, l=0, m=0\rangle$$

In this case $\Delta m = 0$ Hence, at first order in $v_0$ the correction to the ground state energy is zero.

7) The next energy level is $n=2$ and has unperturbed energy $E_2 = -\frac{13.6}{4}$ eV. It has degeneracy 4 and therefore I need to use degenerate perturbation theory.

The levels are $\{|2,0,0\rangle, |2,1,1\rangle, |2,1,0\rangle, |2,1,-1\rangle\}$

I need to compute:

$$\langle 2, l, m | V | 2, l', m' \rangle$$

but I know that most of them are equal to zero! $\uparrow$ $\Delta m \neq 2$ or $-2$

Those that could be different from zero are:

~~$\langle 2\, l\, m | V | 2\, l'\, m' \rangle$~~

$$\langle 2,1,1 | V | 2,1,-1 \rangle \text{ and } \langle 2,1,-1 | V | 2\,1\,1 \rangle.$$

Note that one is the complex conjugate of the other.

$$\cancel{Y_{1,\pm 1} = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi}}$$

The matrix representation of $V_1$ in the basis $\{|200\rangle, |211\rangle, |210\rangle, |21-1\rangle\}$ is:

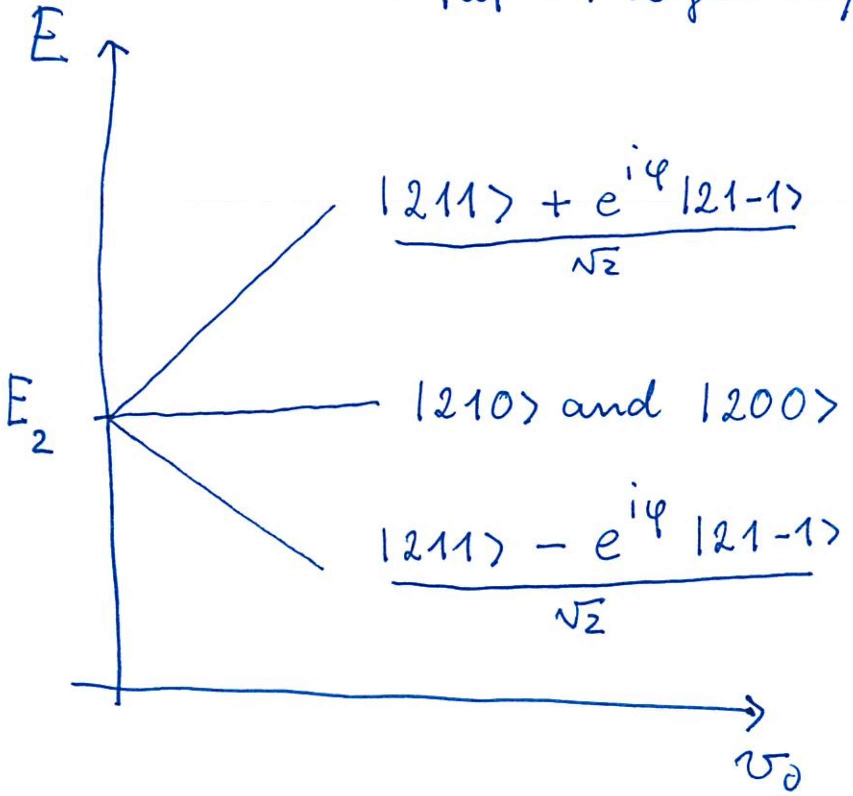
$$V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & a^* & 0 & 0 \end{pmatrix}$$

$$a = \langle 211 | V | 21\,-1\rangle$$

Eigenvalues: $0 \rightarrow$ degeneracy 2

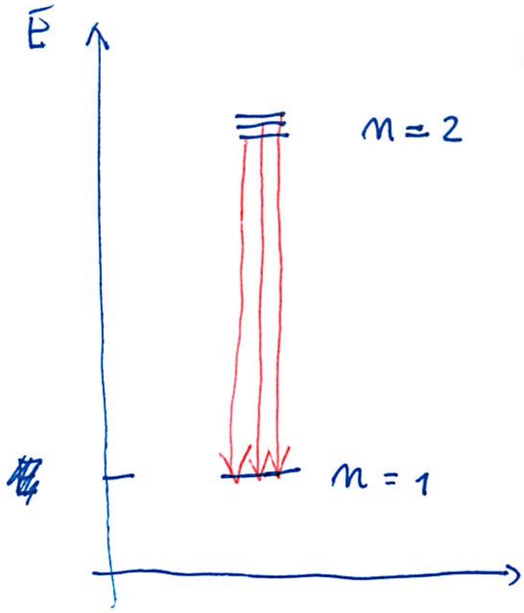
$+|a| \rightarrow$ degeneracy 1

$-|a| \rightarrow$ degeneracy +1



8)

$v_0 \neq 0$




The α-Lyman line is split into three lines.