Exam _ May 2024

Exercise 2

1) Singlet state: $|S=0, M=0\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$ Anti-symmetric

Triplet states: $|S=1, M\rangle = \begin{cases} |\uparrow\uparrow\rangle \\ \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \\ |\downarrow\downarrow\rangle \end{cases}$ Symmetric

2) $g_\ell = 2\ell + 1$, associated with the quantum number m for $\hat{L}_z$.

3) $\vec{r} \longrightarrow -\vec{r}$ for $1 \leftrightarrow 2$. Thus

$$\psi(\vec{r}) = \varphi(r)\, Y_{\ell m}(\theta,\varphi) \xrightarrow{1\leftrightarrow 2} \psi(-\vec{r}) = (-1)^\ell \varphi(r)\, Y_{\ell m}(\theta,\varphi)$$

ℓ even: symmetric under exchange $1 \leftrightarrow 2$

ℓ odd: anti-symmetric under exchange $1 \leftrightarrow 2$

4) $\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2} \longrightarrow \vec{R}$ for $1 \leftrightarrow 2$. Thus

$$\psi(\vec{R}) \xrightarrow{1\leftrightarrow 2} \psi(\vec{R})$$

The properties of the wavefunction are irrelevant because it is always symmetric.

5) Protons are fermions. The total wavefunction should be antisymmetric under $1 \leftrightarrow 2$ exchange

6) Ortho-hydrogen: $S=1$

$\psi(\vec{R})\,\psi(\vec{r})\,|S=1, M\rangle$ should be antisymmetric under exchange $1 \longleftrightarrow 2$

Always symmetric

$\Rightarrow$ $\psi(\vec{r})$ must be anti-symmetric. $\to$ l odd $=1,3,5,7$

7) Para-hydrogen: $S=0$

$\psi(\vec{r})\,\psi(\vec{R})\,|S=0, M=0\rangle$ should be antisymmetric under exchange $1 \longleftrightarrow 2$

always symmetric

always antisymmetric

$\Rightarrow$ $\psi(\vec{r})$ must be symmetric. $\to$ l even $=0,2,4,6\ldots$

8) $$P_{para} = \frac{1}{Z}\sum_{l\ \mathrm{even}} g_l\, e^{-E_l/k_BT}$$ with $$E_l = \frac{\hbar^2}{2I}\,l(l+1)$$

degeneracy in l

$$g_l = (2l+1)$$

$$P_{ortho} = \frac{1}{Z}\sum_{l\ \mathrm{odd}} 3\cdot g_l \cdot e^{-E_l/k_BT}$$

spin-triplet degeneracy.

$$\frac{n_p}{n_o} = \frac{P_{para}}{P_{ortho}} \to \mathrm{OK!}$$

9) $r_* \sim 1\,\text{Å}$ ~ which is more or less $2 \times a_0$, the dimension of the H atom in the ground state $\times 2$.

10) $T = 20\,K$

$$K_B T = 1.38 \cdot 10^{-23} \cdot 20\,J = 27{,}6 \cdot 10^{-23}\,J = 2{,}76 \cdot 10^{-22}\,J$$

$$\hbar = 1.054 \cdot 10^{-34}\,J \cdot s \rightarrow \hbar^2 \sim 1 \cdot 10^{-68}\,J^2 \cdot s^2$$

$$2I = m_p \cdot r_*^2 = 1.67 \cdot 10^{-27}\,Kg \cdot 10^{-20}\,m$$

$$\frac{\hbar^2}{2I} \sim \frac{10^{-68}}{1{,}67 \cdot 10^{-27} \cdot 10^{-20}}\,J \sim 0{,}5\ 10^{-21}\,J \sim 5 \cdot 10^{-22}\,J$$

$K_B T$ is order of $\frac{\hbar^2}{2I}$ .. is the ~~expansion~~ approximation proposed in the text possible? Yes because $x \sim O(1)$ and the exponentiale $e^{-\ell(\ell+1)}$ decays fast. Only the first term of the sum can be retained.

If I perform the expansion proposed by the exercise:

11)

$$\frac{m_p}{m_o} = \frac{1}{3}\,\frac{1}{3 \cdot e^{-2x}} = \frac{1}{9}\,e^{2x} \overset{!}{=} \frac{99.83}{0.17} \cong 587$$

$$e^{2x} \overset{!}{=} 587 \times 9 \cong 5285$$

$$2x = 8{,}57$$

$$x \simeq 4{,}28$$

$$x = \frac{\hbar^2}{2I}\frac{1}{k_B T} \longrightarrow r_*^2 = \frac{\hbar^2}{m_p}\frac{1}{k_B T}\cdot\frac{1}{x}$$

$$= \frac{10^{-68}\ J^2 \cdot s^2}{1.67\cdot 10^{-27}\ kg}\ \frac{1}{2,76\cdot 10^{-22}\ J}\ \frac{1}{4.28}$$

$$\doteq \frac{1}{1,67}\cdot\frac{1}{2,76}\cdot\frac{1}{4,28}\ 10^{-19}\ m$$

$$\simeq 5\cdot 10^{-2}\ m \leftarrow 10^{-21}$$

$$r_* \sim \sqrt{50}\cdot 10^{-1}\ m \sim 7.10^{-11}\ m \qquad \boxed{70\text{pm}}$$

$$\uparrow 10^{-11}$$

which agrees with the value in wikipedia 74.14 pm

(Note that my estimate was of 100 pm)