Exam – May 2024

Exercise 1.

1) $\hat{\sigma}_{1z}\,\hat{\sigma}_{2z}\,|\varepsilon_{1z}\,\varepsilon_{2z}\rangle = (\varepsilon_{1z} \times \varepsilon_{2z})\,|\varepsilon_{1z}\,\varepsilon_{2z}\rangle$

2) We need to invert the relation in equation (1).

$$|+_z\rangle = \langle +_{\vec{u}}|+_z\rangle\,|+_{\vec{u}}\rangle + \langle -_{\vec{u}}|+_z\rangle\,|-_{\vec{u}}\rangle$$

We know from (1): $\langle +_z|+_{\vec{u}}\rangle = \cos\frac{\theta}{2}\,e^{-i\varphi/2}$

$$\langle +_z|-_{\vec{u}}\rangle = -\sin\frac{\theta}{2}\,e^{-i\varphi/2}$$

$$|+_z\rangle = e^{i\varphi/2}\cos\frac{\theta}{2}\,|+_{\vec{u}}\rangle - e^{i\varphi/2}\sin\frac{\theta}{2}\,|-_{\vec{u}}\rangle$$

Next:

$$|-_z\rangle = \langle +_{\vec{u}}|-_z\rangle\,|+_{\vec{u}}\rangle + \langle -_{\vec{u}}|-_z\rangle\,|-_{\vec{u}}\rangle$$

We know from (1): $\langle -_z|+_{\vec{u}}\rangle = \sin\frac{\theta}{2}\,e^{i\varphi/2}$

$$\langle -_z|-_{\vec{u}}\rangle = \cos\frac{\theta}{2}\,e^{i\varphi/2}$$

$$|-_z\rangle = e^{-i\varphi/2}\sin\frac{\theta}{2}\,|+_{\vec{u}}\rangle + e^{-i\varphi/2}\cos\frac{\theta}{2}\,|-_{\vec{u}}\rangle$$

We now consider the state given by the exercise:

$$|+_z -_z\rangle = \cancel{e^{i\varphi_1}} \cos\frac{\theta_1}{2} \sin\frac{\theta_2}{2} |+_{\vec{a}} +_{\vec{b}}\rangle + \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} |+_{\vec{a}} -_{\vec{b}}\rangle +$$

$$- \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} |-_{\vec{a}} +_{\vec{b}}\rangle - \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} |-_{\vec{a}} -_{\vec{b}}\rangle$$

$$|-_z +_z\rangle = \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} |+_{\vec{a}} +_{\vec{b}}\rangle - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} |+_{\vec{a}} -_{\vec{b}}\rangle +$$

$$+ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} |-_{\vec{a}} +_{\vec{b}}\rangle - \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} |-_{\vec{a}} -_{\vec{b}}\rangle$$

Using:

$$\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} = \sin\left(\frac{\theta_2-\theta_1}{2}\right)$$

$$\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} = \cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

$$-\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} - \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} = -\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

$$-\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} = \sin\left(\frac{\theta_2-\theta_1}{2}\right)$$

$$\delta\theta \doteq \frac{\theta_2-\theta_1}{2}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\sin(\delta\theta)\left(|+_{\vec{a}} +_{\vec{b}}\rangle + |-_{\vec{a}} -_{\vec{b}}\rangle\right) +$$

$$+ \frac{1}{\sqrt{2}}\cos(\delta\theta)\left(|+_{\vec{a}} -_{\vec{b}}\rangle - |-_{\vec{a}} +_{\vec{b}}\rangle\right)$$

3.1) $|+_{\vec{a}}\,+_{\vec{b}}\rangle$ $\quad P_{++}=\frac{1}{2}\sin^2\delta\theta$

$|+_{\vec{a}}\,-_{\vec{b}}\rangle$ $\quad P_{+-}=\frac{1}{2}\cos^2\delta\theta$

$|-_{\vec{a}}\,+_{\vec{b}}\rangle$ $\quad P_{-+}=\frac{1}{2}\cos^2\delta\theta$

$|-_{\vec{a}}\,-_{\vec{b}}\rangle$ $\quad P_{--}=\frac{1}{2}\sin^2\delta\theta$

3.2) $|+_{\vec{a}}\rangle$ $\quad P_{++}+P_{+-}=\frac{1}{2}$

$|-_{\vec{a}}\rangle$ $\quad P_{-+}+P_{--}=\frac{1}{2}$

for the first particle.

$|+_{\vec{b}}\rangle$ $\quad P_{++}+P_{-+}=\frac{1}{2}$

$|-_{\vec{b}}\rangle$ $\quad P_{+-}+P_{--}=\frac{1}{2}$

for the second particle.

3.3)

$$P(A|B)=\frac{P_{-+}}{1/2}=\cos^2\delta\theta$$

3.4) $\vec{a}=\vec{b}\rightarrow\delta\theta=0$

$$|\psi\rangle=\frac{1}{\sqrt{2}}\left(|+_{\vec{a}}\,-_{\vec{a}}\rangle-|-_{\vec{a}}\,+_{\vec{a}}\rangle\right)$$

$P_{+-}=1/2 \quad P_{-+}=1/2$ Measurements are anticorrelated.

3.5)

$$E_Q(\vec{a},\vec{b}) = \frac{1}{2}\left(\sin^2\delta\theta\right)\times 2 + \frac{1}{2}\left(\cos^2\delta\theta\right)(-1)\times 2$$

$$= \sin^2\delta\theta - \cos^2\delta\theta = -\cos(2\delta\theta)$$

$$= -\cos(\theta_2-\theta_1)$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cdot\cos(\theta_2-\theta_1)$$

$$= \cos(\theta_2-\theta_1)$$

because $|\vec{a}| = |\vec{b}| = 1$

OK!

$$E_Q(\vec{a},\vec{b}) = -\vec{a}\cdot\vec{b}$$

$E_Q = 1$ means that $\vec{a}$ and $\vec{b}$ are anti-aligned

$$\vec{a} = -\vec{b}.$$

4.1) For any value of $\lambda, \vec{a}, \vec{a}'$, one of $A(\lambda\vec{a}) \pm A(\lambda\vec{a}')$ is equal to zero and the other one to 2.

This term is multiplied by $B(\lambda,\vec{b})$ or $B(\lambda,\vec{b}')$ which is equal to ± 1. Hence:

$$|s(\lambda)| = +2 \quad \forall \lambda, \vec{a}, \vec{b}, \vec{a}', \vec{b}'.$$

4.2)

$$S_c = \int d\lambda\, P(\lambda)\, s(\lambda)$$

$$|S_c| = \left|\int d\lambda\, P(\lambda)\, s(\lambda)\right| \leq \int d\lambda\, \underbrace{|P(\lambda)|}_{=P(\lambda)} \cdot |s(\lambda)| = 2\underbrace{\int d\lambda P(\lambda)}_{1}$$

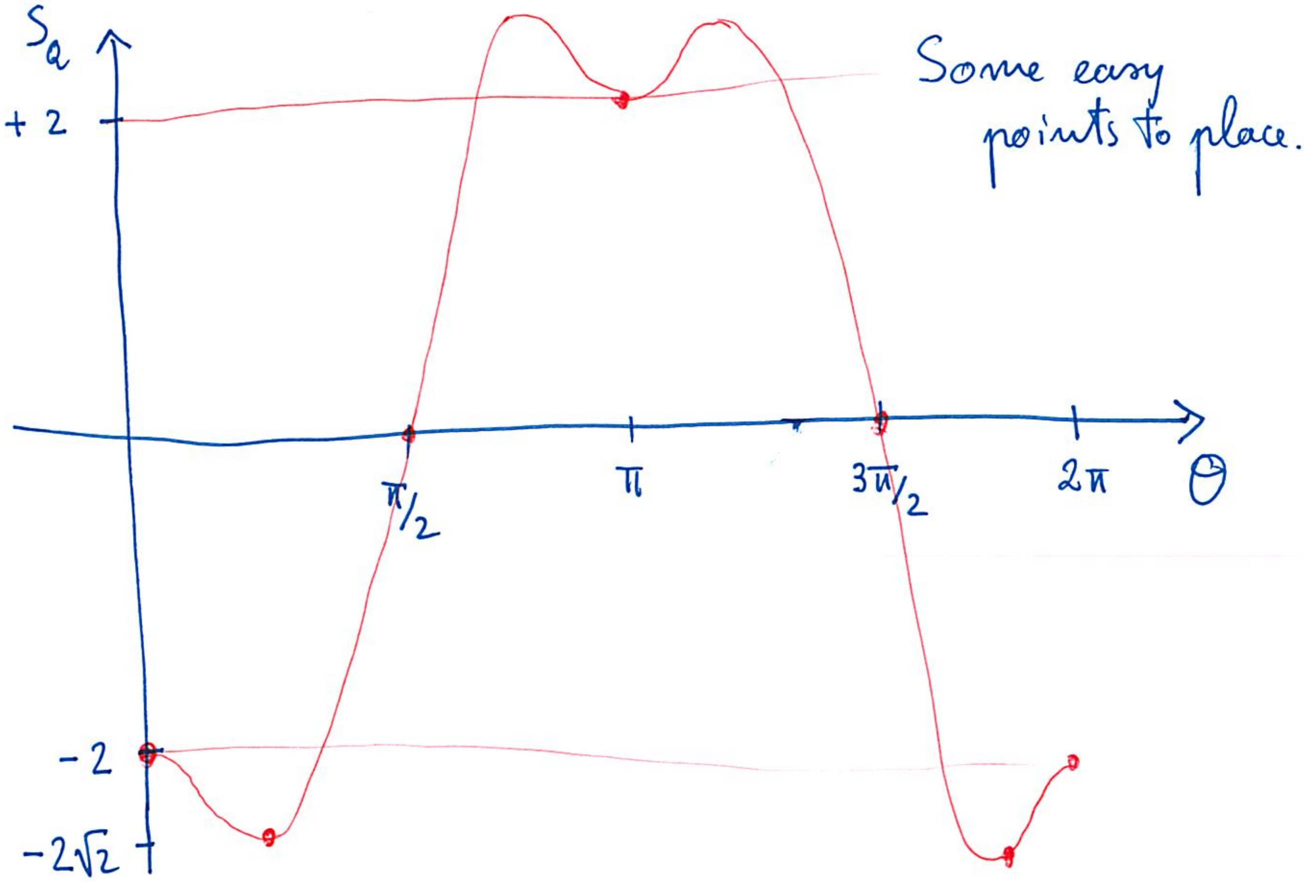
$$\Rightarrow |S_c| \leq 2.$$

5.1

$$S_a = \cancel{4\cos\theta} - \vec{a}\cdot\vec{b}' - \vec{a}'\cdot\vec{b} - \vec{a}'\cdot\vec{b}' + \vec{a}\cdot\vec{b}$$

$$= -3\cos\theta + \cos 3\theta$$

5.2



For $\theta = \frac{\pi}{4}$ $\quad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\quad \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

$S\left(\frac{\pi}{4}\right) = -\frac{4}{\sqrt{2}} < -2$

Seen in class.

$S\left(2\pi - \frac{\pi}{4}\right) = S\left(\frac{\pi}{4}\right)$ because it is a even function.

Centered around $\theta = 0$ and π there are two regions where the Bell inequality is violated