Corrigé – Examen 26 avril 2022 – Leonardo

Exo 1

1.1 – 0,5 pts

- fermions
- antisymmetric with respect to $\hat{P}_{exc}$, the exchange operator

$$\hat{P}_{exc}|\psi\rangle = -|\psi\rangle.$$

1.2 – 0,5 pts

- $$E_{n,m} = \frac{\hbar^2\pi^2}{2mL^2}\left(n^2+m^2\right) \qquad n \geq 1 \quad m \geq 1$$

$$n \geq m$$

1.3 – 1 pts

- Pour $n = m$

$$|\Psi_{n,n}\rangle = |\psi_n\rangle|\psi_n\rangle \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$$

- Pour $n \neq m$

$$|\Psi^{(1)}_{n,m}\rangle = \frac{|\psi_n\rangle|\psi_m\rangle + |\psi_m\rangle|\psi_n\rangle}{\sqrt{2}} \cdot \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$$

$$|\Psi^{(2)}_{n,m}\rangle = \frac{|\psi_n\rangle|\psi_m\rangle - |\psi_m\rangle|\psi_n\rangle}{\sqrt{2}} \cdot \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$$

$$|\Psi^{(3)}_{n,m}\rangle = \frac{|\psi_n\rangle|\psi_m\rangle - |\psi_m\rangle|\psi_n\rangle}{\sqrt{2}} \cdot |\uparrow\uparrow\rangle$$

$$|\Psi^{(4)}_{n,m}\rangle = \frac{|\psi_n\rangle|\psi_m\rangle - |\psi_m\rangle|\psi_n\rangle}{\sqrt{2}} \cdot |\downarrow\downarrow\rangle$$

1.4 – 1 pts

$|\Psi\rangle$ eigenvector of $H \to n = 2,\ m = 1 \to E_{2,1} = \dfrac{\pi^2\hbar^2}{2mL^2}(4+1)$

$|\psi\rangle$ eigenvector of S_z

$$\Rightarrow \quad |\psi_0\rangle = \frac{|\psi_2\rangle|\psi_1\rangle + |\psi_1\rangle|\psi_2\rangle}{\sqrt{2}} \cdot \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \qquad S=0$$

$$|\psi_1\rangle = \frac{|\psi_2\rangle|\psi_1\rangle - |\psi_1\rangle|\psi_2\rangle}{\sqrt{2}} \cdot \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$$

1.5 _ 0,5 points

Both $|\psi_0\rangle$ and $|\psi_1\rangle$ are eigenstates of $\hat{S}^z$ with eigenvalue 0. Hence, if I measure S^z I obtain the value 0 with probability 1. Same result! $\Rightarrow$ I cannot distinguish the states.

1.6 _ 2 points.

Focus just on the spin part.

- $\dfrac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} = \dfrac{|1_+\rangle - |1_-\rangle}{\sqrt{2}}$ hence when I measure S^x I obtain $+\hbar$ with probability $1/2$ and $-\hbar$ with probability $1/2$.

- $\dfrac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} = \dfrac{|0_+\rangle - |0_-\rangle}{\sqrt{2}}$ hence when I measure S^x I obtain 0 with probability 1

$\Rightarrow$ I can distinguish the 2 states!

1.7 _ Bonus 1,5.

A state such that $S^x|\psi\rangle = +\hbar|\psi\rangle$ must be

$$|\psi\rangle = |\rightarrow\rangle_1|\rightarrow\rangle_2 \quad \text{where} \quad |\rightarrow\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$$

Indeed, $|\rightarrow\rangle|\rightarrow\rangle = |1_+\rangle$ after proper expansion.

Similarly, $S^x|\Psi\rangle = -\hbar|\Psi\rangle$ must be $|\Psi\rangle = |\leftarrow\rangle|\leftarrow\rangle$ where $|\leftarrow\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}$. Again, $|\leftarrow\rangle = |1_-\rangle$.

The other two states, $|0_\pm\rangle$ are

$$|0_+\rangle = |\rightarrow\rangle_1 |\leftarrow\rangle_2 \quad \text{and} \quad |0_-\rangle = |\leftarrow\rangle_1 |\rightarrow\rangle_2$$

we just need to do the expansion.

From this writing: $S^x|0_\pm\rangle = (+\hbar/_2 - \hbar/_2)|0_\pm\rangle = 0$.

Exo 2

2.1 — 0,25 pts

- $E_{GS} = E_{000} = \hbar\omega\left(\frac{3}{2}\right) = \frac{3}{2}\hbar\omega$

2.2 — 0,25 pts

- $\psi_{GS}(x,y,z) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-\frac{x^2+y^2+z^2}{2\ell_{HO}^2}}$

2.3 — 0,5 pts

- $\psi_{GS}(r,\theta,\phi) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-r^2/2\ell_{HO}^2}$

No angular dependence: it must be $Y_{00} = \frac{1}{\sqrt{4\pi}}$

$$\psi_{GS}(r,\theta,\phi) = R_{GS}(r)\cdot Y_{00}(\theta,\phi) \quad \text{with}$$

$$R_{GS}(r) = \sqrt{4\pi}\left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-r^2/2\ell_{HO}^2}$$

2.4 — 0,25 pts $L^2 \rightarrow \hbar^2 \ell(\ell+1) \quad \ell = 0,1,2,3\dots$

$L^z \rightarrow \hbar m \quad -\ell \leq m \leq \ell$ AND m integer.

2.5 — 0,5 pts.

$$\begin{cases} L^2 Y_{\ell m}(\theta,\phi) = \hbar^2 \ell(\ell+1) Y_{\ell m}(\theta,\phi) \\ L^z Y_{\ell m}(\theta,\phi) = \hbar m\, Y_{\ell m}(\theta,\phi) \end{cases}$$

Here $\ell = 0$, $m = 0$ $\Rightarrow$ $L^2 \Psi_{GS} = 0 \qquad L^z \Psi_{GS} = 0$

2.6 — 0,5 pts

1st answer: Rotationally invariant wavefunction. No real difference among L_z, L_x, L_y. Hence

$$L_x \Psi_{GS} = 0 \quad \text{AND} \quad L_y \Psi_{GS} = 0$$

2nd answer. $L^2 = L_x^2 + L_y^2 + L_z^2$ sum of positive operators.

If $L^2\Psi = 0$ then $L_x^2 \Psi = 0$
$L_y^2 \Psi = 0$
$L_z^2 \Psi = 0$

Then Ψ is eigenstate of L_α^2 with eigenvalue 0.
It must be eigenvalue of L_α with $\lambda = \pm\sqrt{0} = 0$
Hence $L_\alpha \Psi = 0$

2.7 — 0,5 pts

$$\hat{W} = -\vec{\mu} \cdot \vec{B} = -\gamma B_0 \hat{L}_z$$

2.8 — 0,75 pts

$$\delta E^{(1)}_{GS} = \langle \Psi_{GS} | W | \Psi_{GS} \rangle = -\gamma B_0 \langle \Psi_{GS} | \hat{L}_z | \Psi_{GS} \rangle = 0$$

2.9 — 0,75 pts

$$\delta E_{GS}^{(2)} = \sum_{n > GS} \frac{|\langle n | W | \psi_{GS} \rangle|^2}{E_{GS} - E_n} = \gamma^2 B_0^2 \sum_{n > GS} \frac{|\langle n | \hat{L}_z | \psi_{GS} \rangle|^2}{E_{GS} - E_n} = 0$$

2.10 — 0,5 pts

$$E_1 = \hbar\omega \left(1 + 0 + 0 + \frac{3}{2}\right) = \frac{5}{2} \hbar\omega$$

Three states:

$$\left.\begin{aligned} |\psi_x\rangle &= |1\rangle|0\rangle|0\rangle \\ |\psi_y\rangle &= |0\rangle|1\rangle|0\rangle \\ |\psi_z\rangle &= |0\rangle|0\rangle|1\rangle \end{aligned}\right\} \text{Orthonormal!}$$

Wavefunctions:

$$\psi_x(\vec{r}) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} \frac{\sqrt{2}\, x}{\ell_{HO}}\, e^{-\frac{x^2+y^2+z^2}{2\ell_{HO}^2}}$$

$$\psi_y(\vec{r}) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} \frac{\sqrt{2}\, y}{\ell_{HO}}\, e^{-\frac{x^2+y^2+z^2}{2\ell_{HO}^2}}$$

$$\psi_z(\vec{r}) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} \frac{\sqrt{2}\, z}{\ell_{HO}}\, e^{-\frac{x^2+y^2+z^2}{2\ell_{HO}^2}}$$

2.11 — 0,5 pts

$$V_0(x,y,z) = \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2) = \frac{1}{2} m\omega^2 r^2$$

Central potential: the spectrum is organised in degenerate multiplets with angular momentum ℓ and degeneracy $2\ell + 1$.

Here the degeneracy is 3, hence $\ell = 1$.

Some students could say: three degenerate states with

$l=0$ (accidental degeneracy). This is not the case and mathematical theorems forbid this possibility BUT I suggest to accept the answer as correct.

2.12 — 0,5 pts

$$\hat{L}_z = x p_y - y p_x$$

$$= \frac{l_{HO} p_{HO}}{2} \left((a_x + a_x^\dagger)(-i)(a_y - a_y^\dagger) - (a_y + a_y^\dagger)(-i)(a_x - a_x^\dagger) \right) =$$

$\rightarrow l_{HO} p_{HO} = \hbar$

$$= i \frac{\hbar}{2} \left(a_x a_y - a_x a_y^\dagger + a_x^\dagger a_y - a_x^\dagger a_y^\dagger - a_y a_x + a_y a_x^\dagger - a_y^\dagger a_x + a_x^\dagger a_y^\dagger \right)$$

$$= -i\hbar \left(a_x^\dagger a_y - a_y^\dagger a_x \right) \quad \blacksquare$$

2.13 — 1,0 pts

The state is
$$\psi_z(\vec{r}) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} \frac{\sqrt{2}\, z}{l_{HO}} e^{-\frac{x^2+y^2+z^2}{2 l_{HO}^2}}$$

Using $z = r\cos\theta$ I can pass to spherical coordinates:

$$\psi(r,\theta,\phi) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} \frac{\sqrt{2}}{l_{HO}}\, r\, e^{-r^2/2l_{HO}^2} \cdot \cos\theta.$$

There is one spherical harmonics proportional to $\cos\theta$:

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

Hence:
$$R_2(r) = \sqrt{\frac{4\pi}{3}} \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} \frac{\sqrt{2}}{l_{HO}}\, r\, e^{-r^2/2l_{HO}^2}$$

2.14 — 2 pts

I need to write the restriction of $\hat{L}_z$ on the subspace with energy E_1. We just computed that:

$$L_z |001\rangle = 0$$

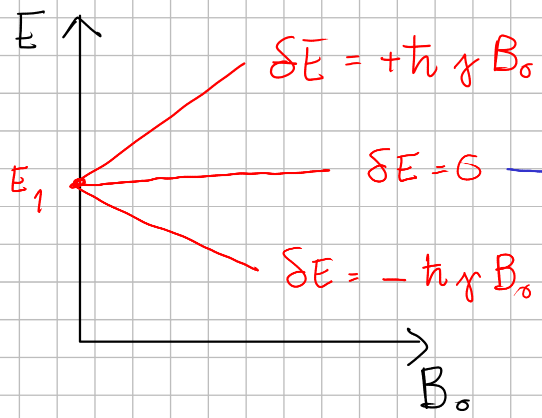
Using the expression given above,

$$L_z |100\rangle = i\hbar |010\rangle$$
$$L_z |010\rangle = -i\hbar |100\rangle$$

Matrix representation in the basis $\{|100\rangle, |010\rangle, |001\rangle\}$

$$W = \begin{pmatrix} 0 & -i\hbar & 0 \\ +i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \text{Eigenvalues: } \hbar, 0, -\hbar$$

Degeneracy completely lifted



$\longrightarrow$ eigenstate of $-\hbar$: $\dfrac{|100\rangle - i|010\rangle}{\sqrt{2}}$

$\longrightarrow$ eigenstate: $|001\rangle$

$\longrightarrow$ eigenstate of $+\hbar$ $\dfrac{|100\rangle + i|010\rangle}{\sqrt{2}}$

From diagon. of the matrix.

2.15 — 1 pts

$$\delta E = \langle \Psi_{GS} | V_1 | \Psi_{GS} \rangle = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int r^2 dr |R_{GS}(r)|^2 \mathcal{U}_{\ell,m}(r) \times$$
$$\times \int \sin\theta \, d\theta \, d\phi \, Y_{00}^* \cdot Y_{\ell m} Y_{00}$$

Since $Y_{00} = \dfrac{1}{\sqrt{4\pi}}$ I can simplify the integral:

$$\int d\Omega\, Y_{00}^* \, Y_{\ell m}\, Y_{00} = \frac{1}{\sqrt{4\pi}} \int Y_{00}^* \, Y_{\ell m}\, d\Omega = \frac{1}{\sqrt{4\pi}} \delta_{\ell,0}\, \delta_{m,0}$$

Finally:

$$\delta E = \frac{1}{\sqrt{4\pi}} \int r^2 dr\, |R_{GS}(r)|^2\, \mathcal{U}_{00}(r)$$

we cannot do more than this because we do not know $\mathcal{U}_{0,0}(r)$.

Exo 3

3.1 – 0,25 pts

$$H = \frac{p^2}{2m} + \frac{1}{2} m \left(\omega_0 + \delta\omega_1(t)\right)^2 x^2$$

$$= \underbrace{\frac{p^2}{2m} + \frac{1}{2} m\, \omega_0^2 x^2}_{H_0} \underbrace{+ \delta\, m\, \omega_0\, \omega_1(t)\, x^2}_{H_1(t)} + \underbrace{\frac{\delta^2}{2} m\, \omega_1(t)^2 x^2}_{H_2(t)}$$

3.2 – 1,0 pts

$$H = H_0 + \delta H_1(t)$$

$$P_{0\to n}(t) = \frac{1}{\hbar^2} \left| \int_0^t e^{-i\omega_{0n} t'} \langle n |\, \delta H_1(t')\, | 0 \rangle\, dt' \right|^2$$

Selection rules: $\langle n | x^2 | 0 \rangle = \frac{\ell_{HO}^2}{2} \langle n | (a + a^\dagger)^2 | 0 \rangle$

$$= \frac{\sqrt{2}}{2} \ell_{HO}^2\, \delta_{n,2}$$

$$P_{0\to n}(t) = \delta^2 \frac{\ell_{HO}^4}{2\hbar^2} m^2 \omega_0^2 \left| \int_0^t e^{-i\omega_{0n} t'}\, \omega_1(t')\, dt' \right|^2$$

$= 1$

3.3 — 2 pts

$$P_{0\to 2}(t) = \frac{\delta^2 \omega_1^2}{2} \left| \int_0^t e^{-i\omega_{02} t'} \sin(\Omega t')\, dt' \right|^2$$

$$\cdot \int_0^t e^{-i\omega_{02} t'} \sin(\Omega t')\, dt' = \frac{1}{2i}\left(\int_0^t e^{-i\omega_{02} t'} e^{i\Omega t'} dt' - \int_0^t e^{-i\omega_{02} t' - i\Omega t'} dt' \right)$$

$$= \frac{1}{2i} \frac{e^{-i(\omega_{02} - \Omega)t} - 1}{-i(\omega_{02} - \Omega)} - \frac{1}{2i} \frac{e^{-i(\omega_{02} + \Omega)t} - 1}{-i(\omega_{02} + \Omega)}$$

A perturbative treatment is possible when $P_{0\to 2}(t) \ll 1$

If we want it to be true at all times

$$\delta\omega_1 \ll |\omega_{02} - \Omega| \quad \text{AND} \quad \delta\omega_1 \ll |\omega_{02} + \Omega|$$

Note that $\omega_{02} = -2\omega_0$

3.4 — 1,5 pts

If I take $\Omega = 2\omega_0$ I get:

$$P_{0\to 2}(t) = \frac{\delta^2 \omega_1^2}{2} \left| \int_0^t e^{-i\omega_{02} t'} \sin(\Omega t')\, dt' \right|^2$$

$$= \frac{\delta^2 \omega_1^2}{8} \left| \int_0^t e^{2i\omega_0 t'} \left(e^{2i\omega_0 t'} - e^{-2i\omega_0 t'} \right) dt' \right|^2$$

$$= \frac{\delta^2 \omega_1^2}{8} \left| -t + \underbrace{\frac{e^{4i\omega_0 t} - 1}{4i\omega_0}}_{\text{Negligible}} \right|^2 \approx \frac{\delta^2 \omega_1^2}{8} t^2$$

Perturbative result valid at short times: $\omega_1 t \ll 1$.