NON LINEAR OPTICS LECTURE 2

Nonlinear Wave Equation

Maxwell's equations

Dielectric Material, free of charges and current

$$\begin{cases} \vec{\nabla} \times \vec{\mathcal{E}} = -\dfrac{\partial \vec{B}}{\partial t} & \vec{\nabla} \cdot \vec{D} = 0 \\ \vec{\nabla} \times \vec{\mathcal{H}} = \dfrac{\partial \vec{D}}{\partial t} & \vec{\nabla} \cdot \vec{B} = 0 \end{cases}$$

with $\vec{B} = \mu_0 \vec{\mathcal{H}}$ and $\vec{D} = \varepsilon_0 \vec{\mathcal{E}} + \vec{P}(t)$

where $\vec{P}(t) = \vec{P}^{(1)} + \vec{P}^{(NL)}(t)$

Non linear response of the material subject to the excited field $\vec{\mathcal{E}}(t)$

$$\Rightarrow \vec{\nabla} \times \vec{\nabla} \times \vec{\mathcal{E}} = -\mu_0 \frac{\partial^2 \vec{D}}{\partial t^2}$$

$$\vec{D} = \varepsilon_0 \vec{\mathcal{E}} + \vec{P}^{(1)} + \vec{P}^{(NL)}(t)$$

$$(1) \quad \boxed{\vec{\nabla} \times \vec{\nabla} \times \vec{\mathcal{E}} + \frac{1}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = -\mu_0 \frac{\partial^2 \vec{P}^{(1)}}{\partial t^2} - \mu_0 \frac{\partial^2 \vec{P}^{(NL)}}{\partial t^2}}$$

Reminder about the Fourier Transform

convention $\quad \vec{\mathcal{E}}(t) = \int \vec{E}(\omega) e^{-i\omega t} d\omega$

$$\hookrightarrow \frac{\partial \mathcal{E}}{\partial t} = \int -i\omega\, E(\omega)\, e^{-i\omega t}\, d\omega$$

$$\frac{\partial^2 \mathcal{E}}{\partial t^2} = \int -\omega^2 E(\omega) \text{———}$$

after
Substitution in (1)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E}(\omega) - \frac{\omega^2}{c^2} \vec{E}(\omega) = +\mu_0 \omega^2 \vec{P}^{(1)}(\omega)$$
$$+ \mu_0 \omega^2 \vec{P}^{(NL)}(\omega)$$

now $\quad P^{(1)}(\omega) = \varepsilon_0\, \underline{\underline{\chi}}^{(1)}(\omega)\, \vec{E}(\omega)$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{\nabla} \times \vec{E}(\omega) - \frac{\omega^2}{c^2} \underline{\underline{\varepsilon}}(\omega)\, \vec{E}(\omega) = \omega^2 \mu_0\, \vec{P}_{NL}(\omega) \qquad (2)}$$

with $\underline{\underline{\varepsilon}} = 1 + \underline{\underline{\chi}}^{(1)}$

$\hookrightarrow$ The relative permittivity

Assumptions

- Material is ~~supposed~~ to be HOMOGENEOUS
+ ISOTROPIC.

$$\varepsilon(\cancel{\vec{x}}, \omega) \longrightarrow \varepsilon(\omega)$$
$$\cancel{\underline{\underline{\varepsilon}}}$$

Taking into account that:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \Delta \vec{E}$$

and that $\vec{\nabla} \cdot \vec{D} = 0 = \vec{\nabla} \cdot (\varepsilon \vec{E}) = 0$

as ε is a scalar and independent of $\vec{r}$

$$\vec{\nabla} \cdot (\varepsilon \vec{E}) = \varepsilon \, \vec{\nabla} \cdot \vec{E} = 0$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \vec{0}}$$

verified only in HOMOGENEOUS + ISOTROPIC material

$$(2) \Rightarrow \boxed{\Delta \vec{E}(\omega) + \frac{\omega^2}{c^2} \varepsilon(\omega) \vec{E}(\omega) = -\omega^2 \mu_0 \vec{P}_{NL}(\omega)} \quad (3)$$

comments

↳ in linear regime, for which $\vec{P}_{NL} = \vec{0}$, we retrieve the well-known linear wave equation

$$\Delta \vec{E} + \frac{\omega^2}{c^2} \varepsilon \vec{E} = 0$$

valid for a linear, isotropic, homogeneous material

↳ in a non linear regime; $\vec{P}_{NL} \neq 0$

plays the role of a DRIVEN term in the differential equation, which in turn is going to modify the field amplitude $\vec{E}(\omega)$

→ Case of a z propagative wave

Assumption:

- Monochromatic wave
- we do not consider the transverse distribution → plane wave.

$\vec{E}(\omega,z) = A(z)\, e^{ikz}\, \vec{e}$ (propagation $z>0$)

↳ Substitution in (3)

$$\boxed{\Delta \vec{E}(\omega) + \frac{\omega^2}{c^2}\varepsilon(\omega)\vec{E}(\omega) = -\omega^2 \mu_0 \vec{P}_{nl}(\omega)} \quad (3)$$

↳ $\Delta = \cancel{\Delta_\perp} + \frac{\partial^2}{\partial z^2}$

$$\vec{e}\cdot\frac{\partial \vec{E}}{\partial z} = \left(\frac{\partial A}{\partial z} + ikA\right)e^{ikz}$$

$$\vec{e}\cdot\frac{\partial^2 \vec{E}}{\partial z^2} = \left(\frac{\partial^2 A}{\partial z^2} + 2ik\frac{\partial A}{\partial z} - k^2 A\right)e^{ikz}$$

$$\Rightarrow \frac{\partial^2 A}{\partial z^2} + 2ik\frac{\partial A}{\partial z} - \underbrace{k^2 A + \frac{\omega^2}{c^2}\varepsilon A}_{=} = -\omega^2\mu_0\, \vec{e}\cdot\vec{P}_{NL}(\omega)\, e^{-ikz}$$

$\Rightarrow$ Dispersion relation $k_{\omega}^2 = \frac{\omega^2}{c^2}\varepsilon(\omega)$

> Slowly varying envelope approximation

$\hookrightarrow \left|\frac{\partial^2 A}{\partial z^2}\right| \ll \left|2k\frac{\partial A}{\partial z}\right|$ with $k = \frac{2\pi n}{\lambda}$

$$\rightsquigarrow \frac{(\Delta A)^2}{\Delta z\,\Delta z} \ll \frac{(\Delta A)^2}{\lambda\,\Delta z}$$

Assuming a very weak perturbation (weak non-linear interaction), one can neglect the variation of the env. A over a typical length $\simeq \lambda$

$\Rightarrow$ Nonlinear wave equation

$$\boxed{\frac{\partial A}{\partial z} = \frac{i\omega}{2nc\varepsilon_0}\,\vec{e}\cdot\vec{P}_{NL}(\omega)\,e^{-ikz}}$$