Invariances and symmetries

2.7 A Heuristic Derivation

We have emphasized that the principles of electromagnetism can only be revealed by experiment. Nevertheless, it is an interesting intellectual exercise to try to deduce the Maxwell equations using only symmetry principles, minimal theoretical assumptions, and (relatively) minimal input from experiment.

The most profound discussions of this type exploit the symmetries of special relativity.[22] Here, we use a heuristic argument based on inversion and rotational symmetry, translational invariance, and four pieces of experimental information: the existence of the Lorentz force, charge conservation, superposition of fields, and the existence of electromagnetic waves.

We begin with experiment and infer the existence of the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$ from trajectory measurements on charged particles. These reveal the Lorentz force,

$$\mathbf{F} = q(\mathbf{E} + \boldsymbol{\upsilon} \times \mathbf{B}). \tag{2.85}$$

We will assume that electromagnetism respects the symmetry operations of rotation and inversion. This means that (2.85) must be unchanged when these orthogonal transformations (Section 1.7) are performed.

Rotational invariance is guaranteed by the fact that $\mathbf{F}$, $\mathbf{E}$, and $\boldsymbol{\upsilon} \times \mathbf{B}$ are all first-rank tensors (vectors). As for inversion, we deduce from the discussion in Section 1.8.1 that $\mathbf{F}$, $\mathbf{E}$, and $\boldsymbol{\upsilon} \times \mathbf{B}$ must all be polar vectors or they must all be axial vectors.[23] The position vector $\mathbf{r}$ is a polar vector. Therefore, since $\boldsymbol{\upsilon} = d\mathbf{r}/dt$ and $\mathbf{F} = d(m\boldsymbol{\upsilon})/dt$, the force $\mathbf{F}$ is also a polar vector. This means that $\mathbf{E}$ and $\boldsymbol{\upsilon} \times \mathbf{B}$ must be polar vectors as well. We have just seen that $\boldsymbol{\upsilon}$ is a polar vector. Hence, (1.162) shows that $\mathbf{B}$ must be an axial vector.

15.2 Symmetry

Symmetry plays many roles in electromagnetism. In Chapters 3 and 10, we used the spatial symmetries of charge and current distributions to help evaluate Coulomb and Biot-Savart integrals. The same symmetries transformed the integral forms of Gauss' law and Ampère's law into useful tools for computation. In Chapters 7 and 8, we used symmetry extensively to choose coordinate systems to separate variables for the Laplace and Poisson equations and to fix values for the separation constants. The last section of Chapter 2 exploited symmetry somewhat differently. There, we *postulated* the invariance of the Maxwell equations to the symmetry operations of spatial translation, rotation, and inversion as part of a heuristic "derivation" of these equations.

Table 15.1. Behavior of electromagnetic quantities under the discrete operations of space inversion (left column) and time inversion (right column).

$\mathbf{r} \to -\mathbf{r}$	$t \to -t$
$\rho \to \rho$	$\rho \to \rho$
$\mathbf{j} \to -\mathbf{j}$	$\mathbf{j} \to -\mathbf{j}$
$\boldsymbol{\upsilon} \to -\boldsymbol{\upsilon}$	$\boldsymbol{\upsilon} \to -\boldsymbol{\upsilon}$
$\mathbf{F} \to -\mathbf{F}$	$\mathbf{F} \to \mathbf{F}$
$\mathbf{E} \to -\mathbf{E}$	$\mathbf{E} \to \mathbf{E}$
$\mathbf{B} \to \mathbf{B}$	$\mathbf{B} \to -\mathbf{B}$
$\varphi \to \varphi$	$\varphi \to \varphi$
$\mathbf{A} \to -\mathbf{A}$	$\mathbf{A} \to -\mathbf{A}$

15.2.1 Discrete Symmetries

A discrete symmetry transformation produces a discontinuous change in a transformed quantity. The discrete symmetries important to electromagnetism are space inversion (or parity), mirror reflection, and time reversal.[1] The operation of space inversion takes the position vector to its opposite: $\mathbf{r} \to -\mathbf{r}$. Other quantities are said to have definite parity under inversion if they either change sign like $\mathbf{r}$ or remain unchanged. Polar vectors and axial vectors are examples of quantities that behave oppositely under inversion (see Application 1.2 at the end of Section 1.8.1).

Following (2.85), we used Newton's second law to establish the polar nature of any force vector and then the Coulomb-Lorentz force $\mathbf{F} = q(\mathbf{E} + \upsilon \times \mathbf{B})$ to fix the parity of the particle velocity $\upsilon = d\mathbf{r}/dt$, the charge density ρ, the current density $\mathbf{j}$, the electric field $\mathbf{E}$, and the magnetic field $\mathbf{B}$. The left column of Table 15.1 summarizes the results of that discussion, together with results for the potentials derived from $\mathbf{E} = -\nabla\varphi$, $\mathbf{B} = \nabla \times \mathbf{A}$, and the rules summarized in (1.162). Reflection through a mirror is similar to inversion in the sense that the relative minus sign between the transformed $\mathbf{r}$ and $\mathbf{B}$ in Table 15.1 applies on a component-by-component basis [compare (10.30) to (10.31)].

The time-reversal operation $t \to -t$ has no effect on $\mathbf{r}$ or the charge density ρ. On the other hand, $\upsilon = d\mathbf{r}/dt$ and hence the linear momentum $\mathbf{p} = m\upsilon$ and current density $\mathbf{j} = \rho\upsilon$ change sign. From this, we conclude that $\mathbf{F} = d\mathbf{p}/dt$ does not change sign under time reversal. Specializing this result to the Coulomb-Lorentz force shows that $\mathbf{E} \to \mathbf{E}$ under time-reversal while $\mathbf{B} \to -\mathbf{B}$.[2] The magnetic field result may be confirmed from the change in magnetic field which occurs when the direction of current flow reverses in any simple configuration of filamentary wires. The behavior of the potentials follows immediately from the behavior of the fields. The right column of Table 15.1 summarizes these results for time reversal.

15.2.3 Continuous Symmetries

A continuous symmetry transformation produces a smooth change in a transformed quantity. Finite changes are regarded as the cumulative effect of a succession of infinitesimal transformations, each of which produces only an infinitesimal change. Familiar continuous symmetries that leave the Maxwell equations invariant include translations in space, rotations in space, and translations in time. Continuous *gauge* transformations of the electromagnetic potentials will occupy our attention in the next section and the continuous *Lorentz transformations* of special relativity are the subject of Chapter 22. For the present, we note only that a homogeneous Lorentz transformation generalizes continuous rotations in three-dimensional space to continuous rotations in four-dimensional space-time.

Continuous symmetries have special interest in theoretical physics because a powerful theorem due to Noether guarantees that each continuous symmetry of a theory generates a conservation law. In the present case, Noether's theorem relates the invariance of the Maxwell equations to translations in space, rotations in space, and translations in time to the conservation laws for linear momentum, angular momentum, and energy, respectively. We delay our discussion of this profound approach to the conservation laws until Chapter 24 when we apply the action principle of Lagrangian mechanics to electrodynamics.