Accordingly, the magnitude of the Poynting vector is

$$S = \frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a} = \frac{VI}{2\pi a L},$$

and it points radially inward. The energy per unit time passing in through the surface of the wire is therefore

$$\int \mathbf{S}\cdot d\mathbf{a} = S(2\pi a L) = VI,$$

which is exactly what we concluded, on much more direct grounds, in Sect. 7.1.1.[4]

Problem 8.1 Calculate the power (energy per unit time) transported down the cables of Ex. 7.13 and Prob. 7.62, assuming the two conductors are held at potential difference V, and carry current I (down one and back up the other).

Problem 8.2 Consider the charging capacitor in Prob. 7.34.

(a) Find the electric and magnetic fields in the gap, as functions of the distance s from the axis and the time t. (Assume the charge is zero at $t = 0$.)

(b) Find the energy density u_{em} and the Poynting vector $\mathbf{S}$ in the gap. Note especially the *direction* of $\mathbf{S}$. Check that Eq. 8.12 is satisfied.

(c) Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the gap (Eq. 8.9—in this case $W = 0$, because there is no charge in the gap). [If you're worried about the fringing fields, do it for a volume of radius $b < a$ well inside the gap.]

8.2 ■ MOMENTUM

8.2.1 ■ Newton's Third Law in Electrodynamics

Imagine a point charge q traveling in along the x axis at a constant speed v. Because it is moving, its electric field is *not* given by Coulomb's law; nevertheless, $\mathbf{E}$ still points radially outward from the instantaneous position of the charge (Fig. 8.2a), as we'll see in Chapter 10. Since, moreover, a moving point charge does not constitute a steady current, its magnetic field is *not* given by the Biot-Savart law. Nevertheless, it's a fact that $\mathbf{B}$ still circles around the axis in a manner suggested by the right-hand rule (Fig. 8.2b); again, the proof will come in Chapter 10.

[4] What about energy flow *down* the wire? For a discussion, see M. K. Harbola, *Am. J. Phys.* **78**, 1203 (2010). For a more sophisticated geometry, see B. S. Davis and L. Kaplan, *Am. J. Phys.* **79**, 1155 (2011).

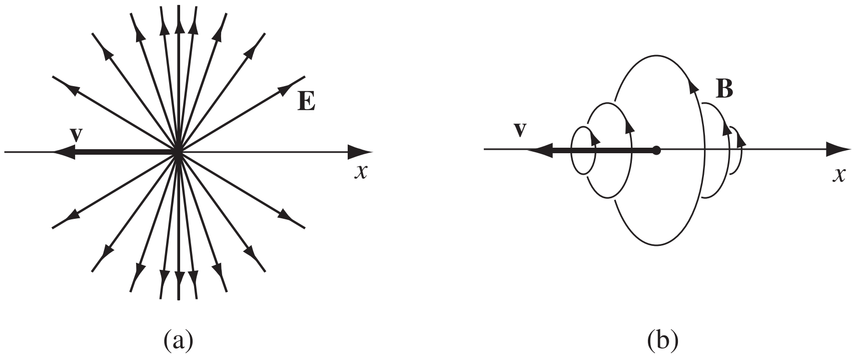

FIGURE 8.2

Now suppose this charge encounters an identical one, proceeding in at the same speed along the y axis. Of course, the electromagnetic force between them would tend to drive them off the axes, but let's assume that they're mounted on tracks, or something, so they're obliged to maintain the same direction and the same speed (Fig. 8.3). The electric force between them is repulsive, but how about the magnetic force? Well, the magnetic field of q_1 points into the page (at the position of q_2), so the magnetic force on q_2 is toward the *right*, whereas the magnetic field of q_2 is *out* of the page (at the position of q_1), and the magnetic force on q_1 is *upward*. *The net electromagnetic force of q_1 on q_2 is equal but not opposite to the force of q_2 on q_1, in violation of Newton's third law*. In electro*statics* and magneto*statics* the third law holds, but in electro*dynamics* it does *not*.

Well, that's an interesting curiosity, but then, how often does one actually use the third law, in practice? *Answer:* All the time! For the proof of conservation of momentum rests on the cancellation of internal forces, which follows from the third law. When you tamper with the third law, you are placing conservation of momentum in jeopardy, and there is hardly any principle in physics more sacred than *that*.

Momentum conservation is rescued, in electrodynamics, by the realization that the *fields themselves carry momentum*. This is not so surprising when you

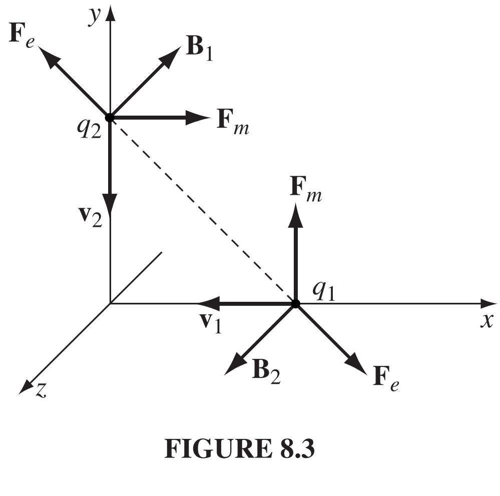

FIGURE 8.3

consider that we have already attributed *energy* to the fields. Whatever momentum is lost to the particles is gained by the fields. Only when the field momentum is added to the mechanical momentum is momentum conservation restored.

8.2.2 ■ Maxwell's Stress Tensor

Let's calculate the total electromagnetic force on the charges in volume $\mathcal{V}$:

$$\mathbf{F} = \int_{\mathcal{V}} (\mathbf{E} + \mathbf{v} \times \mathbf{B})\rho \, d\tau = \int_{\mathcal{V}} (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) \, d\tau. \tag{8.13}$$

The *force per unit volume* is

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}. \tag{8.14}$$

As before, I propose to express this in terms of fields alone, eliminating ρ and $\mathbf{J}$ by using Maxwell's equations (i) and (iv):

$$\mathbf{f} = \epsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} + \left(\frac{1}{\mu_0}\nabla \times \mathbf{B} - \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B}.$$

Now

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \left(\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}\right) + \left(\mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}\right),$$

and Faraday's law says

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E},$$

so

$$\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} = \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times (\nabla \times \mathbf{E}).$$

Thus

$$\mathbf{f} = \epsilon_0 \left[(\nabla \cdot \mathbf{E})\mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})\right] - \frac{1}{\mu_0}\left[\mathbf{B} \times (\nabla \times \mathbf{B})\right] - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}). \tag{8.15}$$

Just to make things look more symmetrical, let's throw in a term $(\nabla \cdot \mathbf{B})\mathbf{B}$; since $\nabla \cdot \mathbf{B} = 0$, this costs us nothing. Meanwhile, product rule 4 says

$$\nabla(E^2) = 2(\mathbf{E} \cdot \nabla)\mathbf{E} + 2\mathbf{E} \times (\nabla \times \mathbf{E}),$$

so

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{2}\nabla(E^2) - (\mathbf{E} \cdot \nabla)\mathbf{E},$$