TD 8

I. 1) $[A+B, C] = (A+B)C - C(A+B) = AC + BC - CA - CB =$

$= [A, C] + [B, C]$

$[AB, C] = ABC - CAB = ABC - ACB + ACB - CAB =$

$= A[B,C] + [A,C]B$

2) by contrapositive: suppose $[A,B] \neq 0$ and $\exists \{\varphi_n\}$ complete orthonormal set

s.t. $A\varphi_n = a_n \varphi_n$ and $B\varphi_n = b_n \varphi_n$.

Then $\forall \psi \in \mathcal{H}$, $\psi = \sum_n c_n \varphi_n$ and:

$[A,B]\psi = (AB - BA)\sum_n c_n \varphi_n = A\left(\sum_n c_n b_n \varphi_n\right) -$

$- B\left(\sum_n c_n a_n \varphi_n\right) =$

$= \sum_n c_n a_n b_n \varphi_n - \sum_n c_n b_n a_n \varphi_n = 0.$

- As it's true $\forall \psi$, $[A,B] = 0$ (contradiction) $\square$

3) using test function $\psi(x)$,

$[X^n, P]\psi = x^n\left(-i\hbar \frac{d\psi}{dx}\right) + i\hbar \frac{d(x^n\psi)}{dx} =$

$= -i\hbar x^n \frac{d\psi}{dx} + i\hbar\left[n x^{n-1} \psi + x^n \frac{d\psi}{dx}\right] = i\hbar n x^{n-1} \psi$

So $[X^n, P] = i\hbar n x^{n-1}$

4) $[A,B] = 0 \Rightarrow AB = BA \Rightarrow A^n B = B A^n$

$e^{-iA} B = \sum_n \frac{(-iA)^n}{n!} B = B \sum_n \frac{(-iA)^n}{n!} = B e^{-iA} \Rightarrow [e^{-iA}, B] = 0$

- As $U(t) = e^{-\frac{i}{\hbar}Ht}$, if $[H,B] = 0$ then we deduce $[U(t), B] = 0$; i.e. B action is unaffected by temporal evolution (more in the following exercises).

II.1

1) $\{A\}$ est ECOC iff there's a <u>unique</u> orthonormal basis such that $A = \sum_k a_k |\varphi_k\rangle\langle\varphi_k|$. So A has to be non degenerate

2) $$p_i' = |\langle\varphi_i'|\psi'\rangle|^2 = |\langle\varphi_i|T^\dagger T|\psi\rangle|^2 = \langle\psi|T^\dagger T|\varphi_i\rangle\langle\varphi_i|T^\dagger T|\psi\rangle = \langle TP_iT\rangle_{T\psi}$$

$$p_i = |\langle\varphi_i|\psi\rangle|^2 = \langle\psi|\varphi_i\rangle\langle\varphi_i|\psi\rangle = \langle P_i\rangle_\psi$$

where $P_i = |\varphi_i\rangle\langle\varphi_i|$ projector on i's eigenspace of A

3) Probabilities have to stay the same after T, so either $T^{-1} = T$ ~~unitary~~ and linear (sends $(T\varphi, T\psi) = (\varphi, \psi)$), either $T^{-1} = T$ and antilinear (sends $(T\varphi, T\psi) = (\psi, \varphi)$). Notably, temporal inversion operator is the latter kind (ANTIUNITARY TRANSFORMATION).

. the result above is further formalized by <u>Wigner's theorem</u>

4) $A \xrightarrow{T} A'$ $\qquad \langle A'\rangle_{\psi'} = \langle A\rangle_\psi$

$$\Rightarrow \langle\psi|T^\dagger A' T|\psi\rangle = \langle\psi|A|\psi\rangle \quad \forall|\psi\rangle \Rightarrow A = T^\dagger A' T$$
$$\Rightarrow \boxed{A' = TAT^\dagger}$$

5) $$[A,T] = AT - TA \underset{A'=A}{=} TAT^\dagger T - TA \underset{T^\dagger T = I}{=} TA - TA = 0$$

6) $$T^\dagger T = I \Rightarrow \left(I + \frac{i}{\hbar}\varepsilon G^\dagger + \ldots\right)\left(I - \frac{i}{\hbar}\varepsilon G + \ldots\right) = I$$
$$\Rightarrow \cancel{I} - \frac{i}{\hbar}\varepsilon G^\dagger + \frac{i}{\hbar}\varepsilon G = \cancel{I} \qquad (\text{ignoring } o(\varepsilon) \text{ terms})$$
$$\Rightarrow \boxed{G^\dagger = G}$$

7) $$A' = TAT^\dagger = \left(I - \frac{i}{\hbar}\varepsilon G + \ldots\right) A \left(I + \frac{i}{\hbar}\varepsilon G + \ldots\right) =$$
$$= A + \frac{i}{\hbar}\varepsilon(AG - GA) = A + \frac{i}{\hbar}\varepsilon[A,G] = A - \frac{i}{\hbar}\varepsilon[G,A]$$

II.2

1) $T^{-1}(a)T(a)|x\rangle = |x\rangle \quad \forall |x\rangle \Rightarrow$

$\Rightarrow T^{-1}(a)|x+a\rangle = |x\rangle \quad , \quad T(-a)|x+a\rangle = |x+a-a\rangle = |x\rangle \quad \forall x$

$\Rightarrow \quad T^{-1}(a) = T(-a) \qquad$ (and $T^{-1} = T^{\dagger}$ from unitarity)

2) $X'|x\rangle = T(a)XT^{\dagger}(a)|x\rangle = T(a)X|x-a\rangle = T(a)(x-a)|x-a\rangle =$

$= (x-a)|x\rangle = X|x\rangle - aI|x\rangle \qquad \forall |x\rangle$

$\Rightarrow \quad X' = X - aI$

- ~~I~~ the analog of this phenomenon is the difference between translating the objects ($T(a)|x\rangle$) and translating the reference frame ($T(a)XT^{\dagger}(a)$). The minus sign shows for consistency between the two pictures

3) We have $\quad X' = X - \frac{i}{\hbar}\varepsilon[G, X] + \dots$

$X' = X - aI \quad$, in particular with $a = \varepsilon$

$\Rightarrow \quad X - \frac{i}{\hbar}\varepsilon[G,X] = X - \varepsilon I \Rightarrow \frac{i}{\hbar}\cancel{\varepsilon}[G,X] = \cancel{\varepsilon}I \Rightarrow$

$\Rightarrow [X, G] = i\hbar \qquad \boxed{G \equiv P}$

4) part. libre $\Rightarrow \quad V(x) = 0 \quad \forall x \quad . \quad H(x) = -\frac{\hbar}{2m}\nabla^2 = \frac{P^2}{2m} = H(x+a)$

$\Rightarrow [T(a), H] = 0 \quad$, and $[T(a), U(t)] = 0 \quad$ (first exercise)

$\frac{d}{dt}\langle\psi(t)|T(a)|\psi(t)\rangle = \frac{d}{dt}\langle\psi(0)|U^{\dagger}(t)T(a)U(t)|\psi(0)\rangle =$

$= \frac{d}{dt}\langle\psi(0)|U^{\dagger}(t)U(t)T(a)|\psi(0)\rangle = \frac{d}{dt}\langle T(a)\rangle_{\psi_0} = 0$

5) $\frac{d}{dt} \langle \psi(t) | T(a) | \psi(t) \rangle = \frac{d}{dt} \langle \psi_0 | (\mathbb{1} - \frac{i}{\hbar} \varepsilon P) | \psi_0 \rangle = 0$

$\Rightarrow \frac{d}{dt} \langle P \rangle_{\psi_0} = 0$, i.e. P is a constant of motion

classical momentum

I (5) (facultatif):

- We define $g(x) = e^{Ax} e^{Bx}$. We try to obtain a differential equation involving $g'(x)$:

$$\frac{dg(x)}{dx} = A e^{Ax} e^{Bx} + e^{Ax} B e^{Bx} =$$
$$= (A + e^{Ax} B e^{-Ax}) e^{Ax} e^{Bx} = (A + e^{Ax} B e^{-Ax}) g(x)$$

- Define $f(x) = e^{Ax} B e^{-Ax}$. Taylor says: $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}\big|_{x=0}$

$$f^{(1)}(x) = e^{Ax} A B e^{-Ax} - e^{Ax} B A e^{-Ax} = e^{Ax} [A, B] e^{-Ax}$$
$$f^{(2)}(x) = e^{Ax} A [A,B] e^{-Ax} - e^{Ax} [A,B] A e^{-Ax} = e^{Ax} [A, [A,B]] e^{-Ax} = 0$$

Hypothesys

$$\Rightarrow f(x) = f(0) + x f^{(1)}\big|_{x=0} = B + x [A, B]$$

- We substitute and obtain: $g'(x) = (A + B + x[A,B])\, g(x)$

that has, as a solution, $e^{xA + xB + \frac{1}{2} x^2 [A,B]}$

- the theorem is obtained with $x = 1$.