# Mathematics for Data Science Exam A

Université Paris-Saclay Informatics Master

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*Answers need to be justified in detail, you can write in English or in French. Books, notes, etc are not allowed. Simple calculators are allowed, but any other electronic devices (mobile phones, etc) are not allowed.*

**Question 1** Given the following vectors and matrices

$$A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 2 & 1 & 0 \\ 2 & 0 & 4 & -2 \\ 5 & -1 & 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$$

- Compute $AB$, $BA$, $AC$, $wv$, $vw$, $B^T$, $C^T$, $\det(A)$, $\det(B)$, $\|v\|_\infty$, $\|v\|_1$, $\|w\|_2$.
- Is $B$ an orthogonal matrix?

**Question 2** Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ and $S\colon \mathbb{R} \to \mathbb{R}$ given by

$$T\colon \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + 2z \\ 0 \\ y + z \end{pmatrix}, \qquad S\colon x \mapsto \begin{cases} 0, & \text{for } x \leq 0, \\ x, & \text{for } x > 0. \end{cases}$$

*The map $S$ is also called ReLU (Rectified Linear Unit) activation function.* Is $T$ a linear transformation? Is $S$ a linear transformation? Justify your answers.

**Question 3** Using Gaussian elimination, solve the following linear system

$$\begin{cases} 2x_1 + 4x_2 + 2x_3 - 2x_4 = 6 \\ 3x_1 + 6x_2 - 6x_4 = 3 \\ x_1 + 2x_2 + 2x_3 = 5 \\ x_1 + 2x_2 + x_3 - x_4 = 3 \end{cases}$$

Moreover, which is the rank of the matrix associated with the linear system?

**Question 4** Let $B$ be a $m \times n$ matrix.

a) Show that $\ker(BB^T) = \ker(B^T)$.

b) Using a), show that $\operatorname{rank}(BB^T) = \operatorname{rank}(B^T)$.

**Question 5** Here we denote the 2-norm $\| \; \|_2$ by $\| \; \|$. Show that for all $x, y \in \mathbb{R}^n$ we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

**Question 6** Prove that the determinant of an orthogonal matrix is equal to $\pm 1$. Prove that the absolute value of the determinant of a square matrix $A$ is equal to the product of the singular values of $A$.