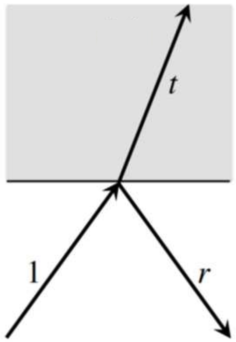
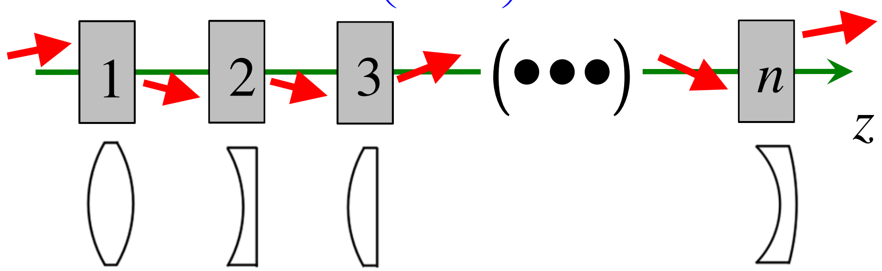
Bonjour!

Points to remember Lecture 2:

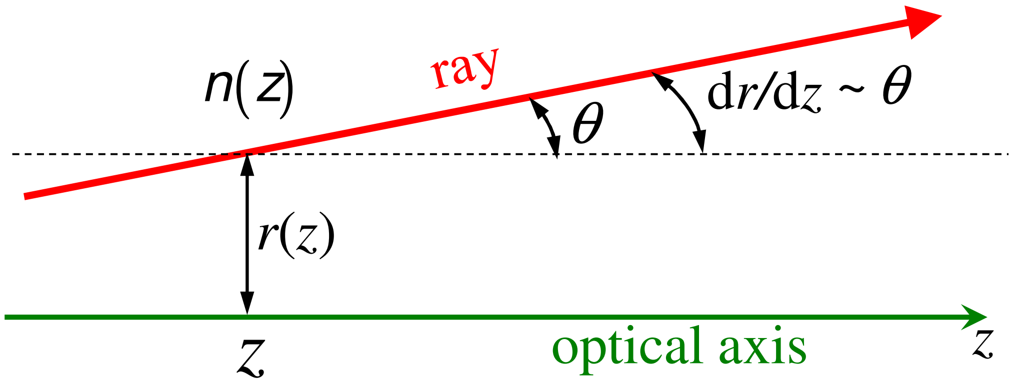


- Fresnel's equations and applications: Fabry-Perot interferometer, Bragg mirror

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathbf{M}_n \mathbf{M}_{n-1} \ldots \mathbf{M}_2 \mathbf{M}_1$$



- Matrix optics

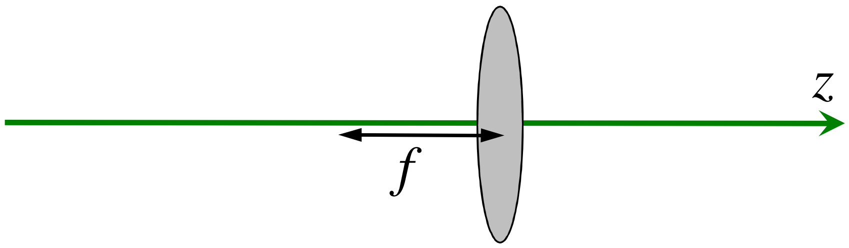


Ray vector: $\mathbf{R}(z) \equiv \begin{pmatrix} r(z) \\ r'(z) \end{pmatrix}$

Reduced slope

$$r'(z) \equiv n(z) \frac{\mathrm{d}r}{\mathrm{d}z}$$

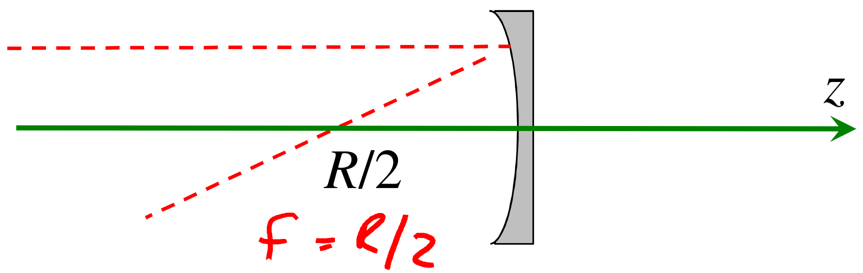
Matrix optics: thin lens of focal length f in air



$$M_{lens} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{pmatrix}$$

$f > 0$ for a convergent lens

Matrix optics: sphercial mirror of focal length f



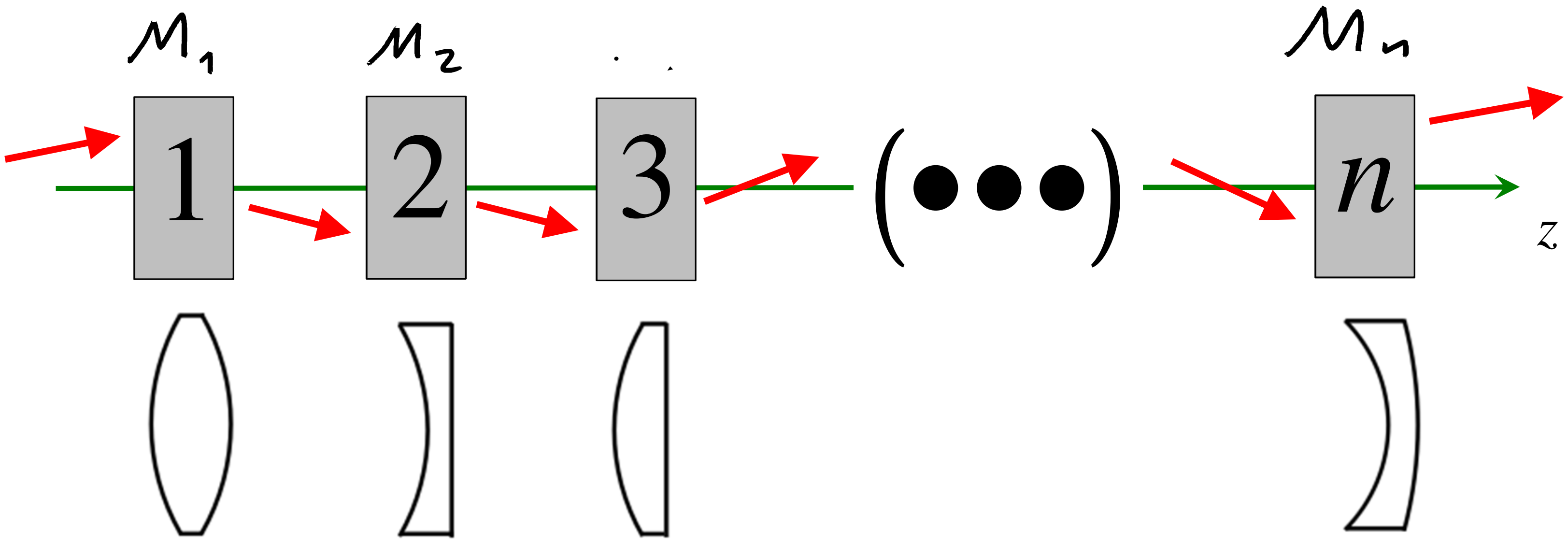
$R > 0$ for a concave mirror

$$M_{spherical\_mirror} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}$$

Si pas d'absorption $\quad \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = 1$

$AD - BC = 1$

Matrix optics and an optical system



$$\boxed{M_{total} = M_n M_{n-1} \cdots M_1}$$

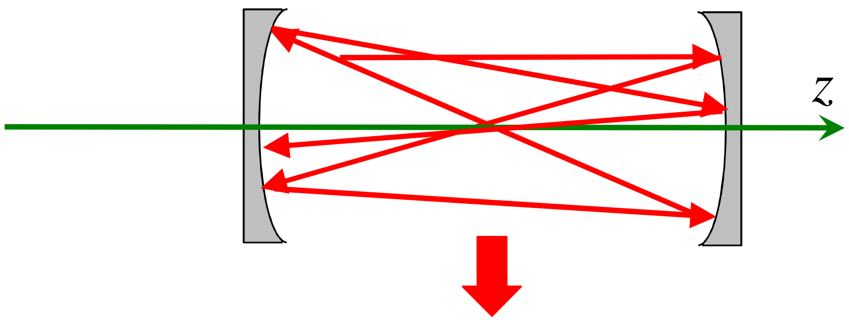
Vote 1 : $M_n \cdots M_1$

Vote 2 : $M_1 \cdots M_n$

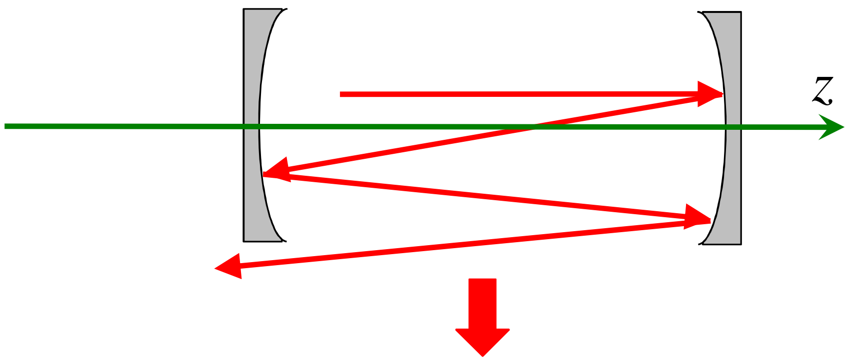
Attention à l'ordre!

Matrix optics and cavity stabiliy

géométrique

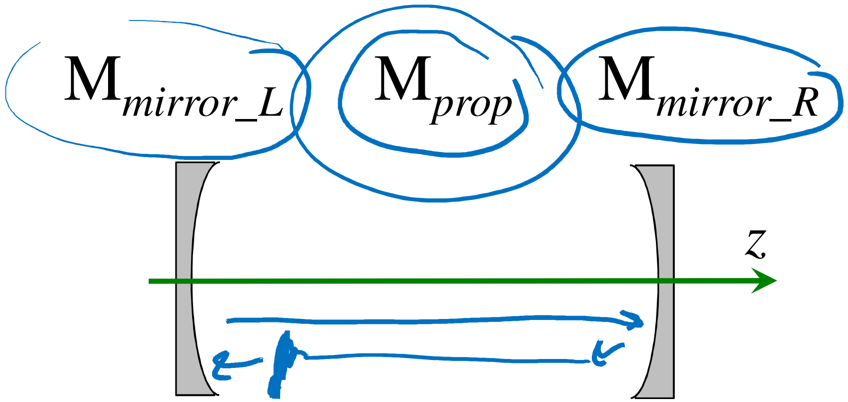


STABLE CAVITY!



UNSTABLE CAVITY!

Cavity as a series of periodic elements



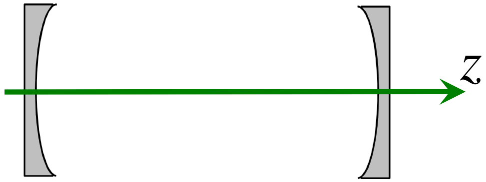
M_{RT}
aller-retour

=

etc . . . M_{prop} | M_{mirror_L} | M_{prop} | M_{mirror_R} | M_{prop} | M_{mirror_L}

$\boxed{\text{1 period = 1 round trip}}$ ⇒ $\mathbf{M}_{RT}$

Plan of attack for investigating cavity stability



$$\boxed{\vec{r}_n = \underline{M}_{RT}^{\,n}\,\vec{r}_0}$$

1. Find eigenvalues of (general) round-trip matrix $\mathbf{M}_{RT} \Rightarrow \boxed{\lambda_\pm}$

2. Find corresponding eigenvectors $\Rightarrow \boxed{\mathbf{r}_+}, \boxed{\mathbf{r}_-}$

3. Express initial ray $\boxed{\mathbf{r}_0}$ in terms of these eigenvectors

4. Find an expression for $\boxed{\mathbf{r}_n}$ in terms of these eigenvectors and eigenvalues and examine it.

Find eigenvalues

1. Find eigenvalues of (general) round-trip matrix $\mathbf{M}_{RT} \Rightarrow \boxed{\lambda_{\pm}}$

$$\mathbf{M}_{RT} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \det\begin{pmatrix} A-\lambda & B \\ C & D-\lambda \end{pmatrix} = 0 \qquad AD - BC = 1$$

$$\lambda_{\pm} = m \pm \sqrt{m^2 - 1} \text{ with } m = \frac{A+D}{2}$$

Cavity stability: steps 2 to 4

$M_{RT}^n \vec{r}_+ = \lambda_+^n \vec{r}_+$

2. Eigenvectors: $\mathbf{M}_{RT}\mathbf{r}_+ = \lambda_+\mathbf{r}_+$
$\mathbf{M}_{RT}\mathbf{r}_- = \lambda_-\mathbf{r}_-$

3. Let $\mathbf{r}_0 = a\mathbf{r}_+ + b\mathbf{r}_-$

4. $\boxed{\mathbf{r}_n} = ?$ $\vec{r}_n = M_{RT}^n \vec{r}_0 = M_{RT}^n a \vec{r}_+ + M_{RT}^n b \vec{r}_-$

$= \lambda_+^n a \vec{r}_+ + \lambda_-^n b \vec{r}_-$

Condition for stability?

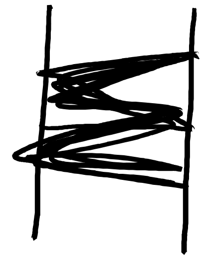
$\boxed{|\lambda_\pm| < 1}$

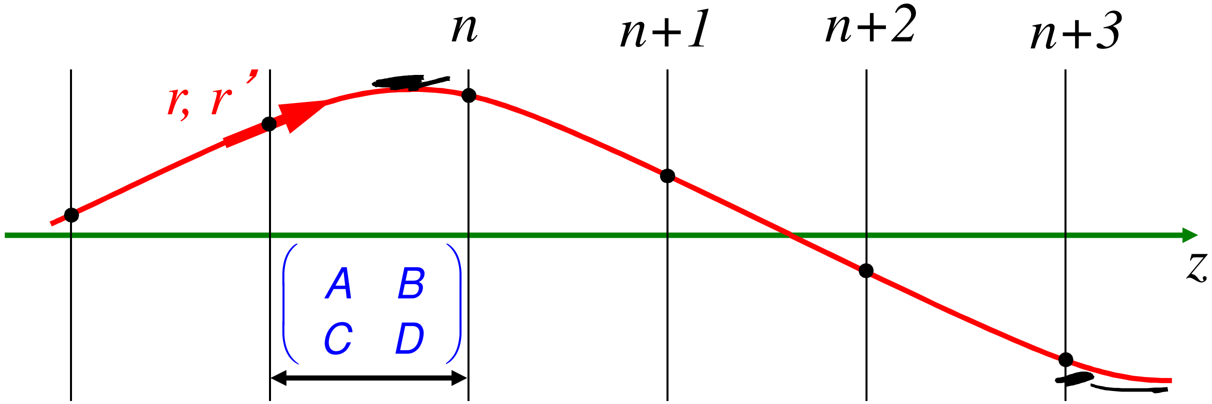
Recall: $\lambda_\pm = m \pm \sqrt{m^2 - 1}$ with $m = \dfrac{A+D}{2}$

$\Rightarrow -1 \le m \le 1$

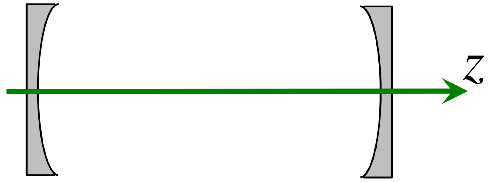
$-1 \le \dfrac{A+D}{2} \le 1$

Stable cavity





$$\mathbf{r}_n = ae^{in\theta}\mathbf{r}_+ + be^{-in\theta}\mathbf{r}_- = \mathbf{r}_0 \cos n\theta + \mathbf{s}_0 \sin n\theta$$



Unstable cavity

$$|\lambda_+| > 1 \text{ or } |\lambda_-| > 1$$

$$\mathbf{r}_n = \mathbf{M}_{RT}^n \mathbf{r}_0 = \mathbf{M}_{RT}^n \left(a\mathbf{r}_+ + b\mathbf{r}_-\right) = a\lambda_+^n \mathbf{r}_+ + b\lambda_-^n \mathbf{r}_-$$

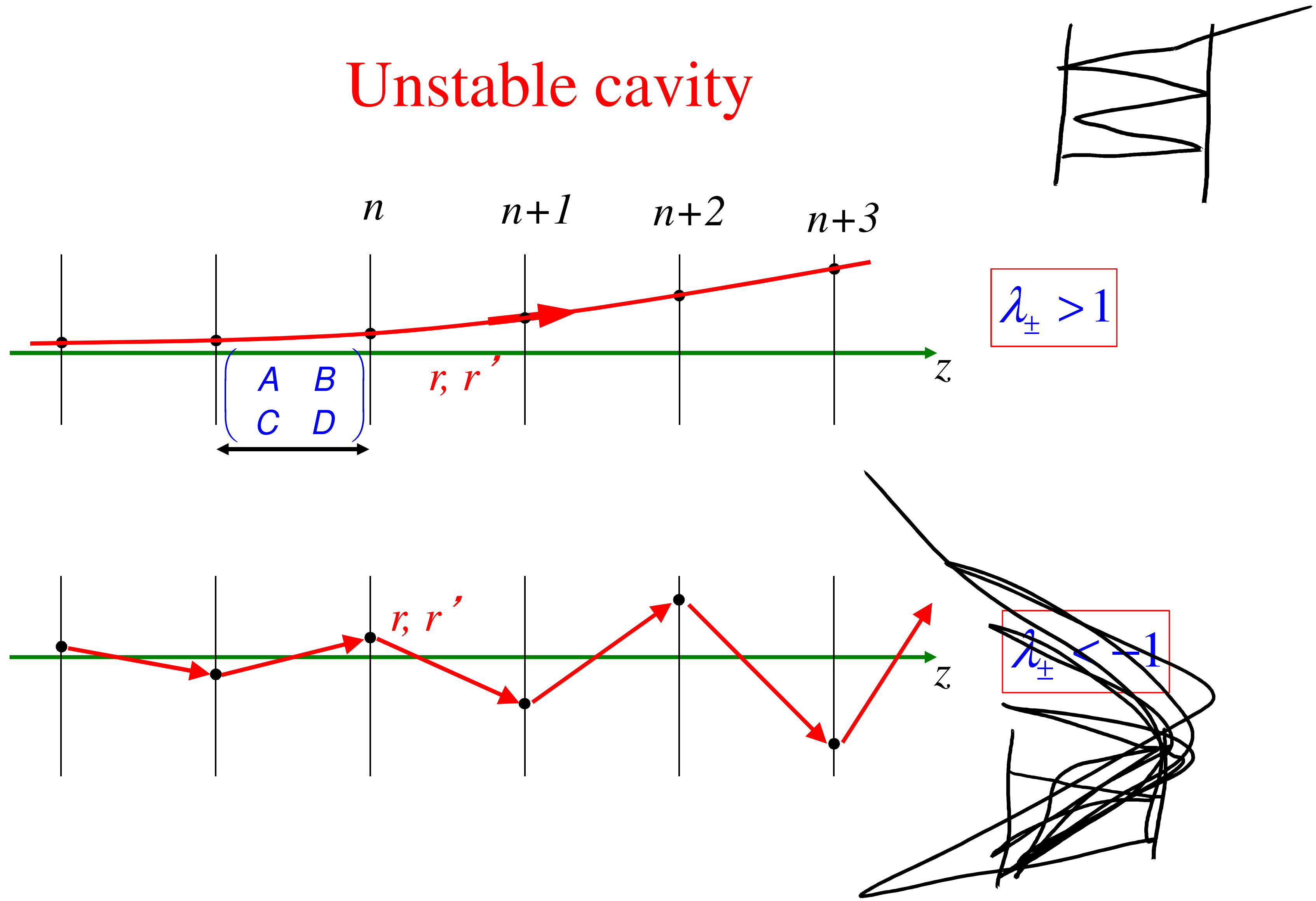
$$\lambda_\pm = m \pm \sqrt{m^2 - 1} \text{ with } m = \frac{A+D}{2} \qquad |m| > 1$$

Let $\quad m = \frac{A+D}{2} = \pm\cosh\theta$

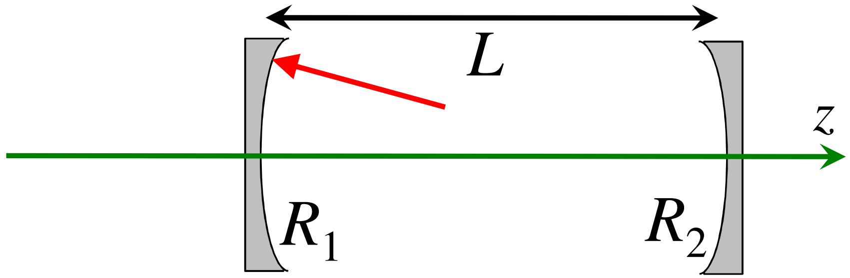
$$\lambda_\pm = \cosh\theta \pm \sqrt{\cosh^2\theta - 1} = \cosh\theta \pm \sinh\theta = e^{\pm\theta}$$

$$\mathbf{r}_n = a\lambda_+^n \mathbf{r}_+ + b\lambda_-^n \mathbf{r}_- = a\left(\pm e^{\theta}\right)^n \mathbf{r}_+ + b\left(\pm e^{-\theta}\right)^n \mathbf{r}_-$$

Unstable cavity



Example: Cavity with two spherical mirrors



$$\mathbf{M}_{prop} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{M}_{mirror\_i} = \begin{pmatrix} 1 & 0 \\ -\dfrac{2}{R_i} & 1 \end{pmatrix}$$

$$\mathbf{M}_{RT} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} =$$

$$\mathbf{M}_{RT} = \begin{pmatrix} \left(1-\dfrac{2L}{R_2}\right)\left(1-\dfrac{2L}{R_1}\right)-\dfrac{2L}{R_1} & L\left(1-\dfrac{2L}{R_2}\right)+L \\ -\dfrac{2}{R_2}\left(1-\dfrac{2L}{R_1}\right)-\dfrac{2}{R_2} & -\dfrac{2L}{R_2}+1 \end{pmatrix}$$

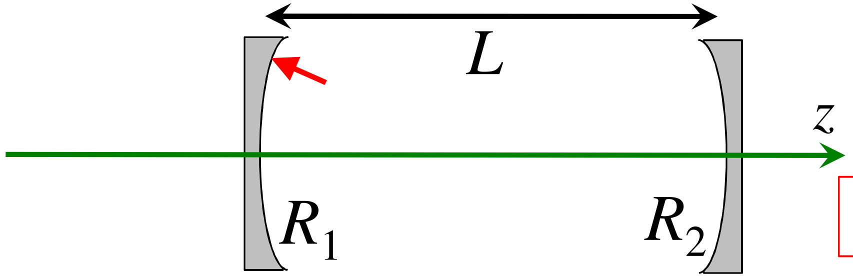
$$m = \frac{A+D}{2} = 2\underbrace{\left[1-\frac{L}{R_1}\right]}_{\equiv g_1}\underbrace{\left[1-\frac{L}{R_2}\right]}_{\equiv g_2}-1 \equiv 2g_1g_2-1$$

cavité stable

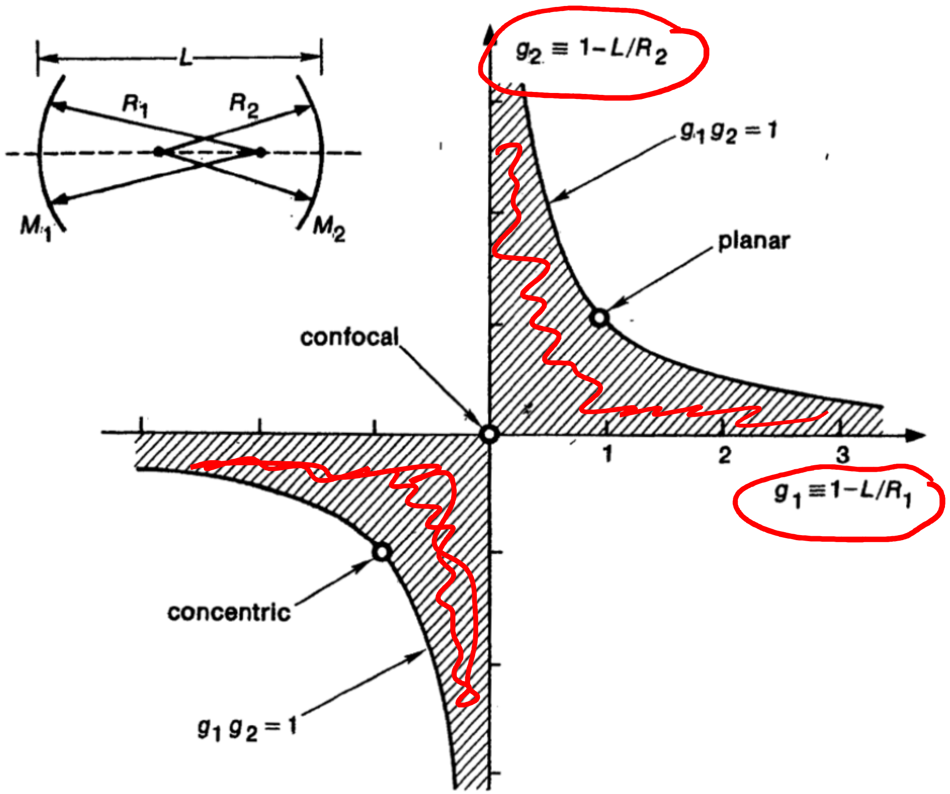
⇨ $-1 \le m \le 1$

$$\boxed{\begin{array}{c}\text{Cavité stable} \\ 0 \le g_1 g_2 \le 1\end{array}}$$

Example: Cavity with two spherical mirrors



Stable cavity $\Longleftrightarrow$ $0 \leq g_1 g_2 \leq 1$



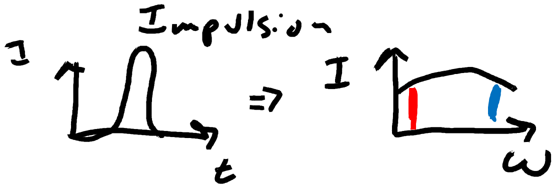
Pulse propagation in Dispersive media

couleur = fréquence

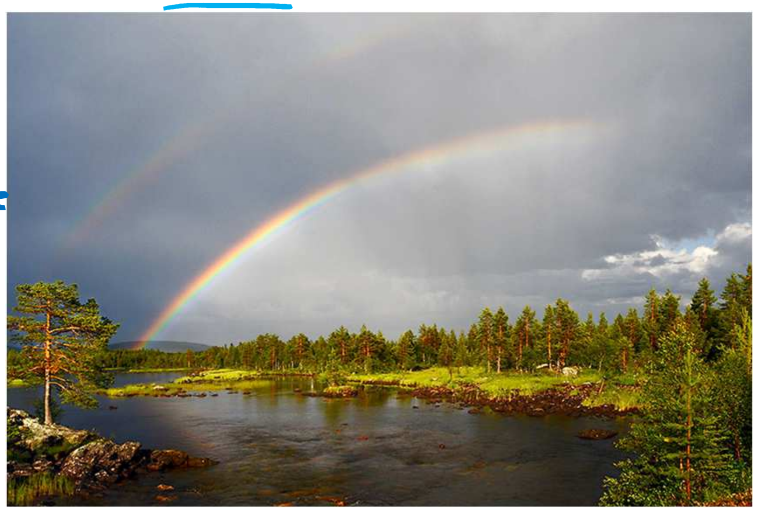
$n = n(\omega)$, $\mu = \mu(\omega)$

$\chi_e = \chi_e(\omega)$ $\quad \epsilon = (\omega)$ ....

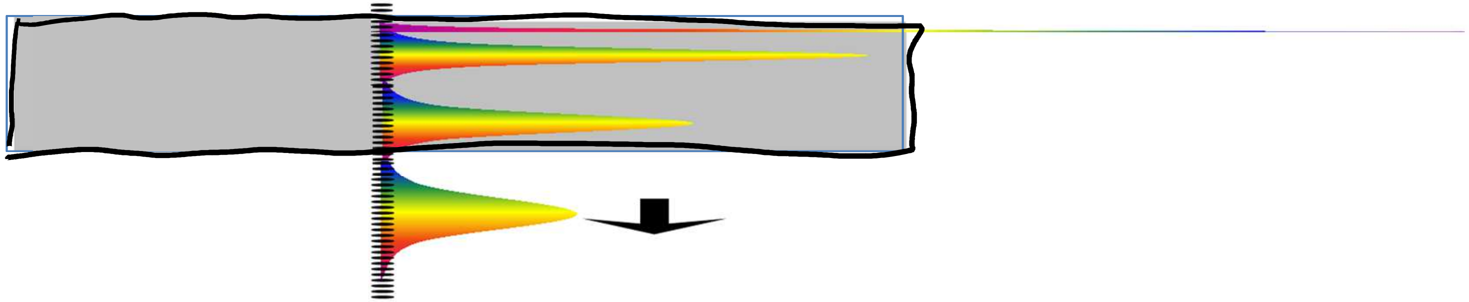
⇒ Les propriétés optiques dépendent de la fréquence du rayonnement



$$\boxed{V = \frac{c}{n(\omega)}}$$



$n = n(\omega)$



Goals for lecture 3

- Dispersion
  - Where does it come from?
  - What are its consequences?
- Propagation of pulses or wave packets in dispersive media

Introduction

- Dispersion occurs since

  a medium cannot respond instantaneously

  to an electromagnetic wave.

- The response of a material to an EM field must be

  causal i.e., it can depend on values of the field

  that existed in the past

  but not on those that will exist in the future!

$n(\omega)$ → per-tes

- Consequence: frequency dispersion and energy dissipation

  are intimately related.

What is the origin of dispersion?

If we accept the electromagnetic theory of light, there is nothing left but to look for the cause of dispersion in the molecules of the medium itself.

Hendrik Lorentz (1878)
Nobel prize (1902)

Convolution:

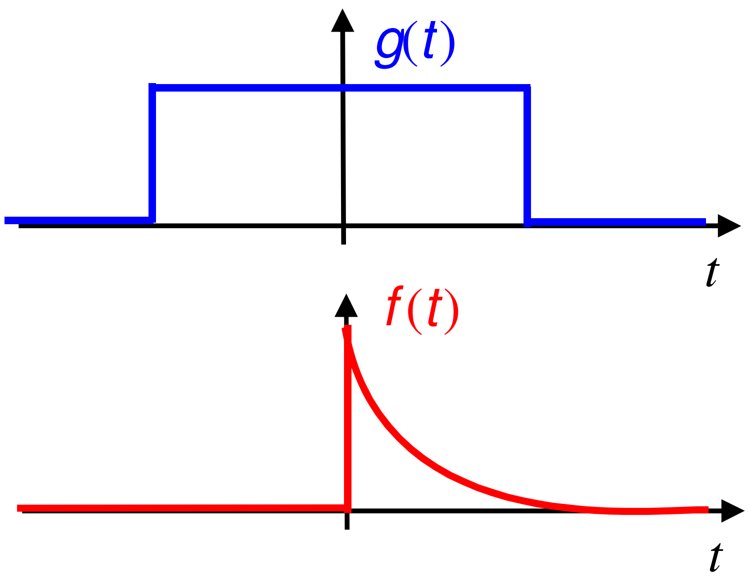
$\circledast$

$$\left[f(t) \circledast g(t) \right](t) = \int_{-\infty}^{\infty} d\tau \, f(t-\tau)\, g(\tau)$$

Système linéaire

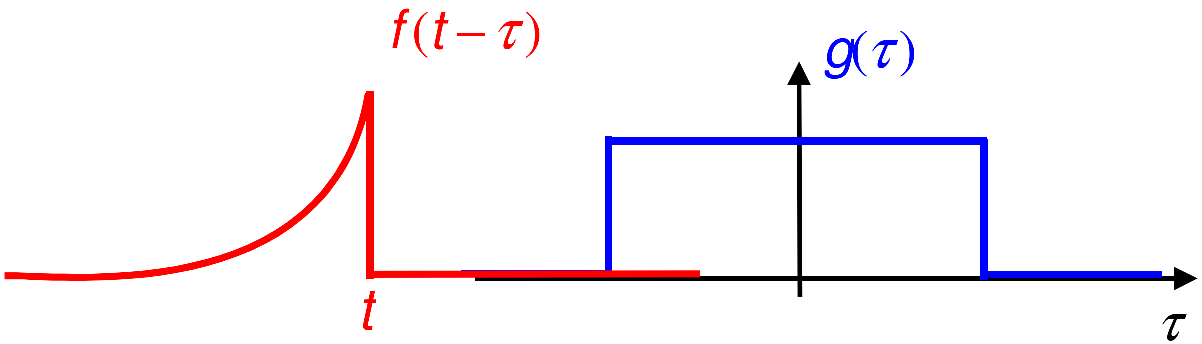
$g(t)$: excitation

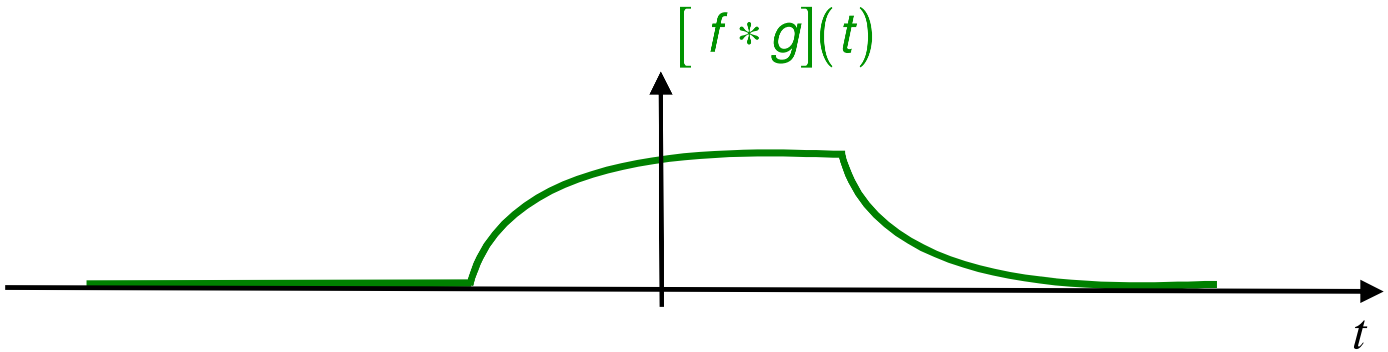
$f(t)$: (non-instantaneous) impulse response function of the system



Convolution: excitation + system response (linear system)

$$\boxed{\left[f(t) * g(t) \right](t) = \int_{-\infty}^{\infty} d\tau \, f(t-\tau)\, g(\tau)}$$





What is the origin of dispersion?

$\vec{j} = \frac{dI}{da_{\perp}}$ densité de courant

$\vec{E} \rightarrow g$ "excitation"

Example: Conductivity $\mathbf{j} = \sigma \mathbf{E}$

$\sigma = \frac{1}{\rho}$

↑ conductivité $\quad \rho \leftarrow$ résistivité

$\sigma \Rightarrow f$ "réponse du système à une impulsion"

$$\boxed{\mathbf{j}(\mathbf{r},t) = \int_{-\infty}^{t} dt' \, \sigma(t-t')\, \mathbf{E}(\mathbf{r},t')} \qquad (4.1)$$

Causality: can only depend on values of the field that existed in the $\boxed{\text{past}}$

Let $\tau \equiv t - t'$

Build causality into the conductivity! Define

$$\sigma(\tau) = 0 \text{ for } \tau < 0$$

$\sigma(\tau) \rightarrow 0$ for $\tau \rightarrow \infty$ Distant past has no influence.

$$\Rightarrow \quad \mathbf{j}(\mathbf{r},t) = \int_{-\infty}^{\infty} dt' \, \sigma(t-t')\, \mathbf{E}(\mathbf{r},t')$$

→ convolution!

Fourier analysis

https://commons.wikimedia.org/wiki/File:Fourier_transform_time_and_frequency_domains.gif

Fourier analysis for non-periodic functions

$\vec{E} = \vec{\mathcal{E}}_0 \, e^{i\vec{k}\cdot\vec{r} - i\omega t} + c.c.$

espace <=> fréquence spatiale

$x \Leftrightarrow k_x$

$\vec{r} \Leftrightarrow \vec{k}$

temps <=> fréquence temporelle

$(\omega = 2\pi f$

$t \Leftrightarrow \omega)$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)e^{ikx}dk$$

$$f(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, f(\omega)\, e^{-i\omega t}$$

$$f(\omega) = \int_{-\infty}^{\infty} dt\, f(t)\, e^{i\omega t}$$

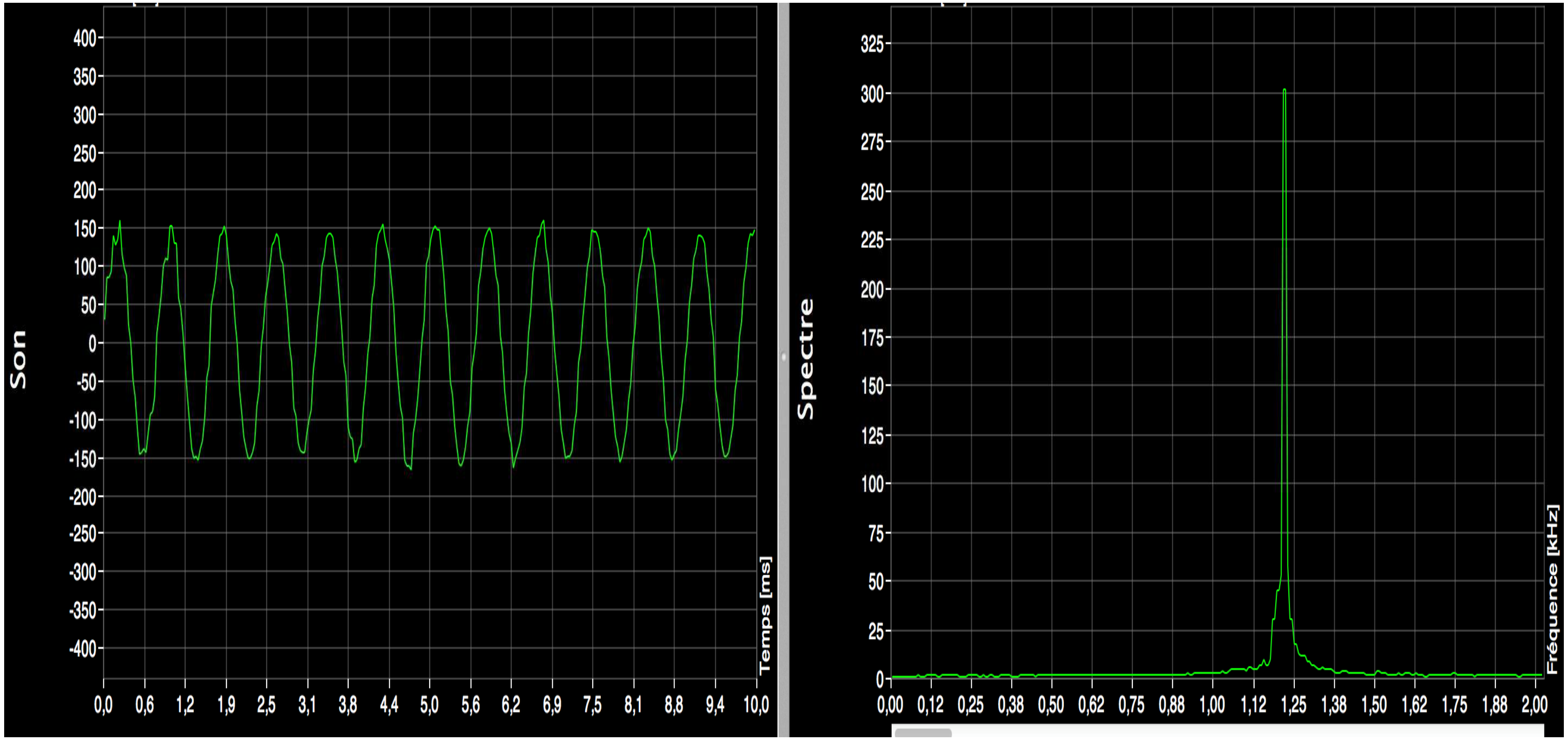
Fourier transformations

Note: signs and position of 1/(2π) is a matter of convention.

Fourier transforms

A physical interpretation

$|F(\omega)|^2$ is the power spectral density of the function $F(t)$, i.e., it tells us how much power is present at each frequency



Some properties of Fourier transforms

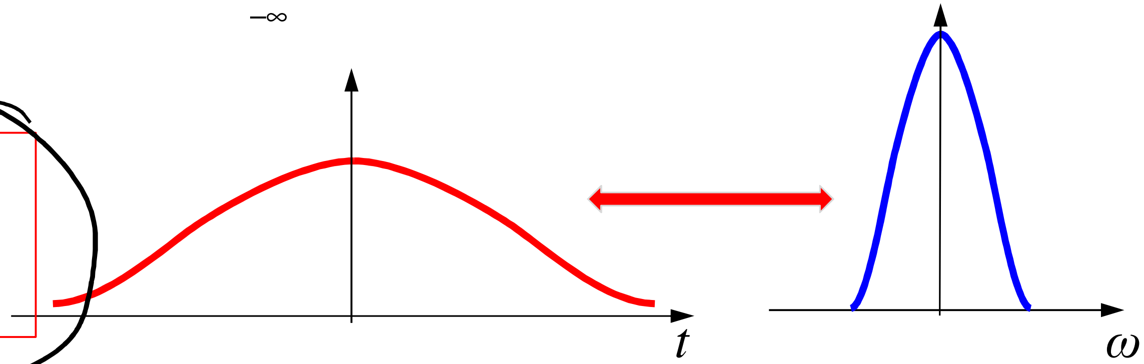
Widths

Suppose that for $F(t)$ and $F(\omega)$: $\int_{-\infty}^{\infty} F(t)\,dt = 0 \quad and \quad \int_{-\infty}^{\infty} F(\omega)\,d\omega = 0$

Suppose that $F(t)$ is normalized: $\int_{-\infty}^{\infty} |F(t)|^2\,dt = 1$

Then: $\boxed{\Delta t \Delta\omega \geq \pi}$

The larger $F(t)$ the narrower $F(\omega)$ and vice versa



PARSEVAL-PLANCHEREL theorem:

$$\int_{-\infty}^{\infty} F_1(t)\left[F_2(t)\right]^* dt = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} F_1(\omega)\left[F_2(\omega)\right]^* d\omega$$

Some properties of Fourier transforms

Derivatives and the Fourier transform

$$\frac{d}{dt}F(t) \rightleftharpoons -i\omega F(\omega)$$

$$itF(t) \rightleftharpoons \frac{d}{d\omega}F(\omega)$$

Fourier transform of a real function:

$$F(t) \in \mathbb{R} \Rightarrow F(-\omega) = \left[F(\omega)\right]^*$$

Some properties of Fourier transforms

$$F(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right) \longleftrightarrow F(\omega) = \sqrt{2\pi}\sigma \exp\left(-\frac{\omega^2\sigma^2}{2}\right)$$

Gaussian

$$\left(\frac{d}{dt}\right)^n F(t) \longleftrightarrow (-i\omega)^n F(\omega)$$

$$t^n F(t) \longleftrightarrow \left(-i\frac{d}{d\omega}\right)^n F(\omega)$$

$$F(at) \longleftrightarrow \frac{1}{|a|} F(\omega / a)$$

$$F(t+t_0) \longleftrightarrow e^{-i\omega t_0} F(\omega)$$

$$e^{i\omega_0 t} F(t) \longleftrightarrow F(\omega+\omega_0)$$

Reciprocal spaces

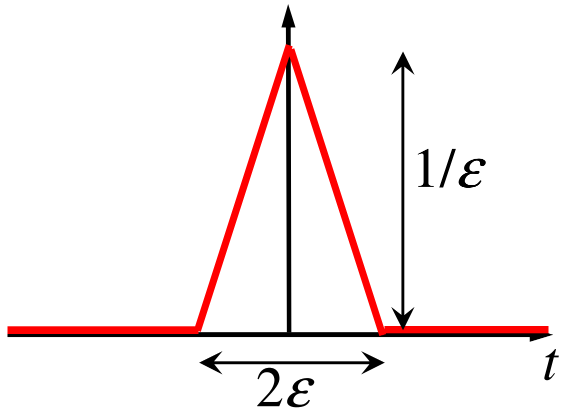
Time $\Longleftrightarrow$ (**temporal**) frequency

Position $\Longleftrightarrow$ **spatial** frequency

Miguel Covarrubias,
http://www.loc.gov/pictures/item
/acd1996002431/PP/

Dirac delta function

$$\delta(t)$$



$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

Area = 1 $\quad \displaystyle\int_{-\infty}^{\infty} \delta(t)dt = 1$

$$\delta(t) = \delta(-t)$$

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt = f(t_0)$$

Fourier transform:

$$1 = \int_{-\infty}^{\infty} \delta(t) e^{i\omega t}\ dt$$

$$\delta(t) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} e^{-i\omega t}\ d\omega$$

Recall: Fourier transforms and the convolution integral

In general:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\; f(\omega)\, e^{-i\omega t}$$

$$f(\omega) = \int_{-\infty}^{\infty} dt\; f(t)\, e^{i\omega t}$$

$$\left[f(t) * g(t) \right](t) = \int_{-\infty}^{\infty} d\tau\; f(t-\tau)\, g(\tau) \quad\longleftrightarrow\quad f(\omega)\cdot g(\omega)$$

multiplication

"The Fourier transform of a convolution integral is equal to the product of the Fourier transforms of the individual functions".

$$2\pi f(t)\cdot g(t) \quad\longleftrightarrow\quad \left[f(\omega) * g(\omega) \right](\omega) = \int_{-\infty}^{\infty} d\omega'\; f(\omega-\omega')\, g(\omega')$$

Fourier transform pairs

$$\hat{\sigma}(\omega) = \int_{-\infty}^{\infty} \sigma(t) e^{i\omega t} dt$$

(complexe)

$$\sigma(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\sigma}(\omega) e^{-i\omega t} d\omega$$

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r},\omega) e^{-i\omega t} d\omega$$

Selon ce qu'on a comme argument, on sait s'il s'agit de la fonction ou de sa TF.

$$\mathbf{j}(\mathbf{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{j}(\mathbf{r},\omega) e^{-i\omega t} d\omega$$

Non-instantaneous response of a dispersive medium

$$\boxed{\mathbf{j}(\mathbf{r},t) = \int_{-\infty}^{\infty} dt' \, \sigma(t-t') \mathbf{E}(\mathbf{r},t')}$$

Fourier transform

$$\boxed{\mathbf{j}(\mathbf{r},\omega) = \hat{\sigma}(\omega)\mathbf{E}(\mathbf{r},\omega)}$$

$\sigma' \in \mathbb{R}$

$\sigma'' \in \mathbb{R}$

$$\hat{\sigma}(\omega) = \underline{\sigma'}(\omega) + i\underline{\sigma''}(\omega)$$

What can you say about the symmetry of the real and imaginary parts of σ?

Properties of the real and imaginary contributions

Recall: $\sigma(t)$ is real $\longrightarrow$ $\sigma(t) = \sigma^*(t)$

From the definition of the Fourier transform:

$\omega \to -\omega$

$$\sigma(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\sigma'(\omega) + i\sigma''(\omega)]\, e^{-i\omega t}\, d\omega = \sigma^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\sigma'(\omega) - i\sigma''(\omega)]\, e^{+i\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sigma'(-\omega) - i\sigma''(-\omega) e^{-i\omega t}\, d\omega$$

$\Rightarrow$ réelle $\sigma'(\omega) = \sigma'(-\omega) \Rightarrow$ PAIRE $\Rightarrow n, \chi, \mu, \epsilon \ldots$

imaginaire $\sigma''(\omega) = -\sigma''(-\omega) \Rightarrow$ IMPAIRE

Change ω to $-\omega$ in the second integral and conclude!

Non-instantaneous response of a dispersive medium

$$\boxed{\sigma(\tau) \in \mathbb{R} \Rightarrow \hat{\sigma}(-\omega) = \hat{\sigma}^*(\omega) \Rightarrow \begin{cases} \sigma'(\omega) & \text{even} \\ \sigma''(\omega) & \text{odd} \end{cases}}$$

Same conclusions for $\hat{\sigma}(\omega), \hat{\mu}(\omega), \hat{\chi}(\omega), \hat{\varepsilon}(\omega)$:
All are complex and depend on the frequency.

And what if dispersion didn't exist???

What happens if the response is the same for all frequences?

i.e., $\boxed{\hat{\sigma}(\omega) \equiv \hat{\sigma}}$ – const

(inverse) Fourier transform

$\int_{-\infty}^{\infty} e^{-i\omega t} \, d\omega$

Instantaneous response--impossible! Frequency dispersion MUST exist

$$\sigma(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\sigma} e^{-i\omega t} d\omega = \frac{\hat{\sigma}}{2\pi} \underbrace{\int_{-\infty}^{\infty} e^{-i\omega t} \, d\omega}_{2\pi\delta(t)} = \hat{\sigma}\,\delta(t)$$

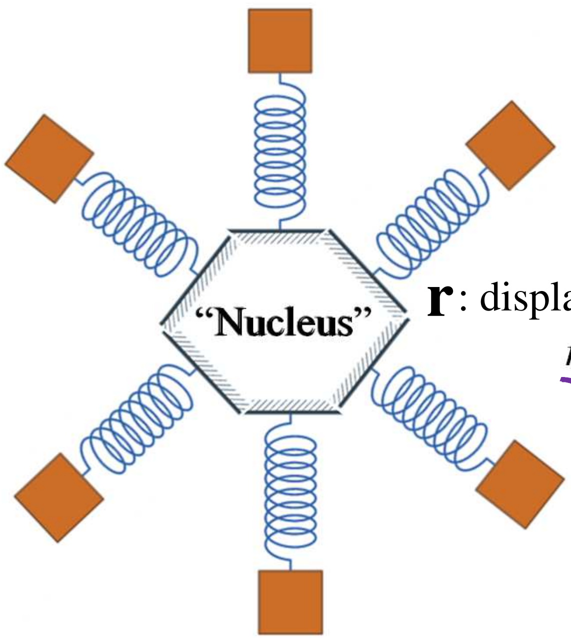
INSTANTANEOUS RESPONSE OF THE SYSTEM

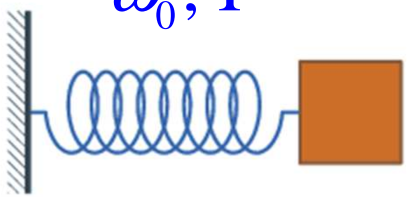
Next: Classical models for frequency dispersion: the Lorentz model

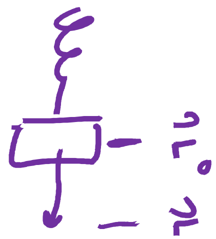
Trouver $n(\omega)$ pour un diélectrique (isolant)

Basic idea: the movement of the (bound) electrons in a material

is that of a *damped, driven, simple harmonic oscillator*







$\mathbf{r}$: displacement from equilibrium

m: electron mass

ω_0 resonance frequency

Γ damping constant

$\vec{F} = m\vec{a}$

amortissement

force de rappel

$$m\frac{d^2\mathbf{r}}{dt^2} + m\Gamma\frac{d\mathbf{r}}{dt} + m\omega_0^2\mathbf{r} = -e\mathbf{E}$$

force champ $\vec{E}$

And for "free" electrons? ➡ No spring! ➡ $\boxed{\text{Drude model for metals!!!}}$

Lorentz Model

$$\boxed{m\frac{d^2\mathbf{r}}{dt^2}+m\Gamma\frac{d\mathbf{r}}{dt}+m\omega_0^2\mathbf{r}=-e\mathbf{E}} \qquad (4.45)$$

Consider a steady state regime:

Monochromatic field $\quad \mathbf{E}(t)=\mathcal{E}\exp(-i\omega t)+\text{c.c.} \qquad (4.46)$

Expression for displacement: $\quad \mathbf{r}(t)=\mathcal{R}\exp(-i\omega t)+\text{c.c.} \qquad (4.47)$

Plug (4.46) and (4.47) in (4.45) and solve for $\mathcal{R}$!

$$-m\omega^2\vec{\mathcal{R}}-i\omega\Gamma\vec{\mathcal{R}}+m\omega_0^2\vec{\mathcal{R}}=-e\vec{\mathcal{E}}$$

q $+$ $-$ $\quad \vec{r}$

$\vec{p}$: dipôle

$\vec{p}=q\vec{r}$

$$\boxed{\mathcal{R}=\frac{-e/m}{\omega_0^2-\omega^2-i\omega\Gamma}\mathcal{E}}$$

Dipole moment:

$$\boxed{\mathbf{p}=-e\mathbf{r}(t)=\frac{e^2/m}{\omega_0^2-\omega^2-i\omega\Gamma}\mathcal{E}\exp(-i\omega t)+\text{c.c.}}$$

Lorentz Model

$n^2(\omega) = \frac{\epsilon(\omega)}{\epsilon_0}$

$\vec{P}$: moment dipolaire par unité de volume

Dipole moment: $\mathbf{p} = -e\mathbf{r}(t) = \frac{e^2/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}\mathcal{E}\exp(-i\omega t) + \text{c.c.}$

Polarization (electric dipole moment per unit volume):

$$\mathbf{P} = \tilde{n}\mathbf{p}(t) = \frac{\tilde{n}e^2/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}\mathcal{E}\exp(-i\omega t) + \text{c.c.} \qquad (4.49)$$

$\tilde{n}$: number of electrons per unit volume

liés

Recall: $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon\mathbf{E}$ Use (4.49) in this expression Let $\omega_p^2 = \frac{\tilde{n}e^2}{\varepsilon_0 m}$

fréquence plasma

$\epsilon_0 \vec{E} + \frac{\epsilon_0 \omega_p^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}\vec{E} = \epsilon \vec{E}$

Solve for $n^2(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_0}$

This n is the index of refraction!

$$\boxed{n^2(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}}$$

modèle de Lorentz

Lorentz model

Complex index of refraction

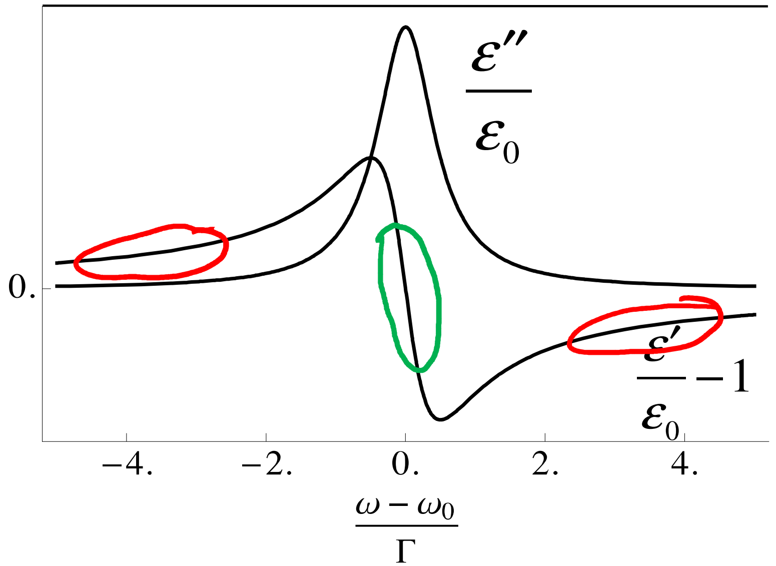
$$n^2(\omega) = \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}$$

$$\omega_p^2 = \frac{\tilde{n}e^2}{\varepsilon_0 m}$$

$\frac{dn}{d\omega} > 0$ dispersion "normal"

$\frac{dn}{d\omega} < 0$ dispersion "anormal"

Real and imaginary parts



n ~ relativement const. loin d'une résonance

$$\frac{\varepsilon'(\omega)}{\varepsilon_0} = 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}$$

$$\frac{\varepsilon''(\omega)}{\varepsilon_0} = \frac{\omega_p^2\omega\Gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}$$

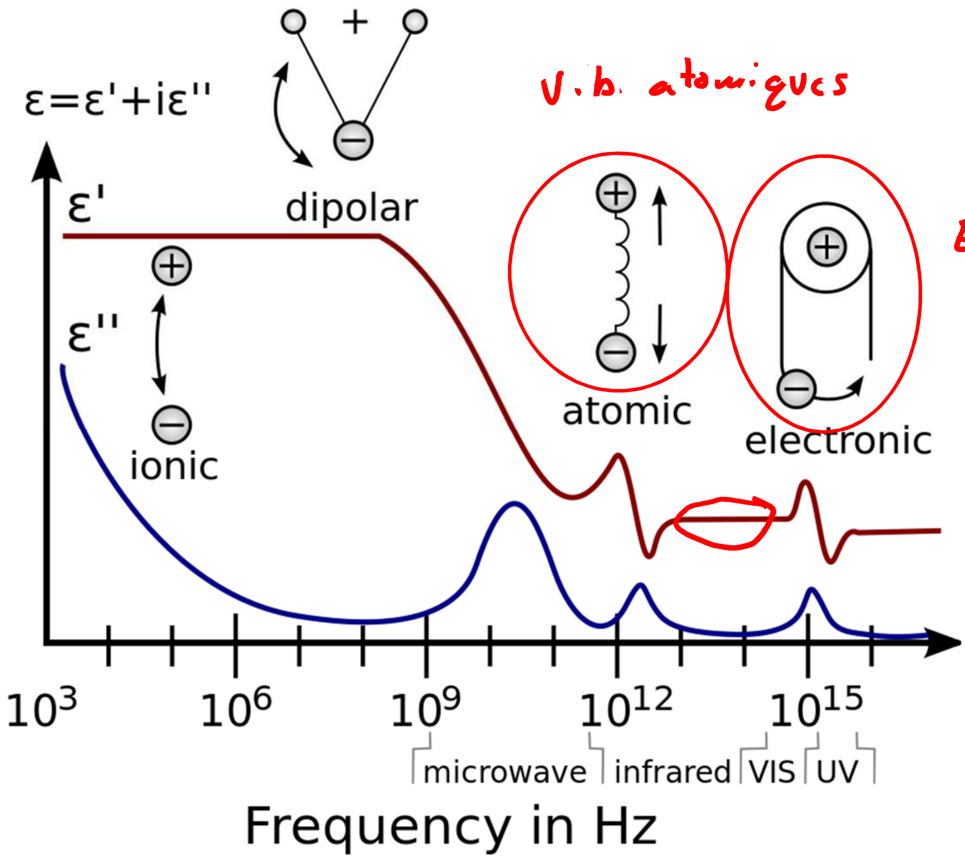
Almost Lorentzian if $\Gamma \ll \omega_0$

Dielectrics

Frequency dependence of the permittivity

vib. atomiques

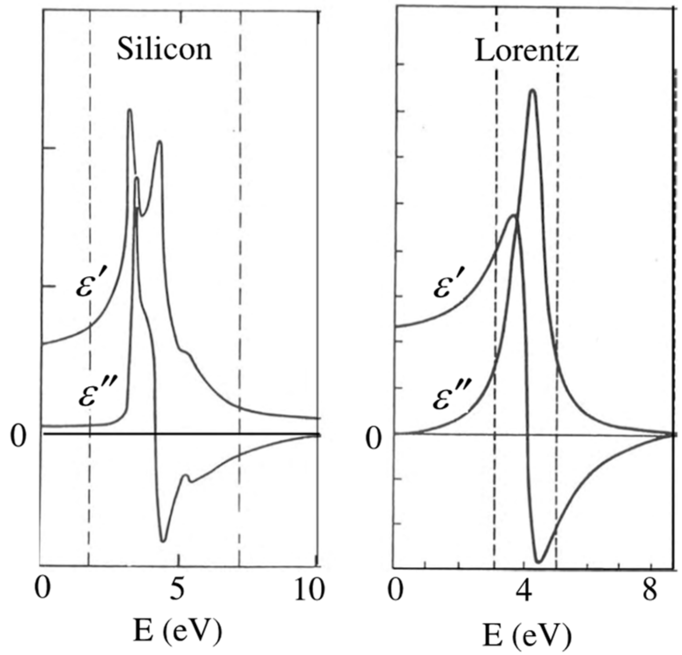
Excitations électroniques

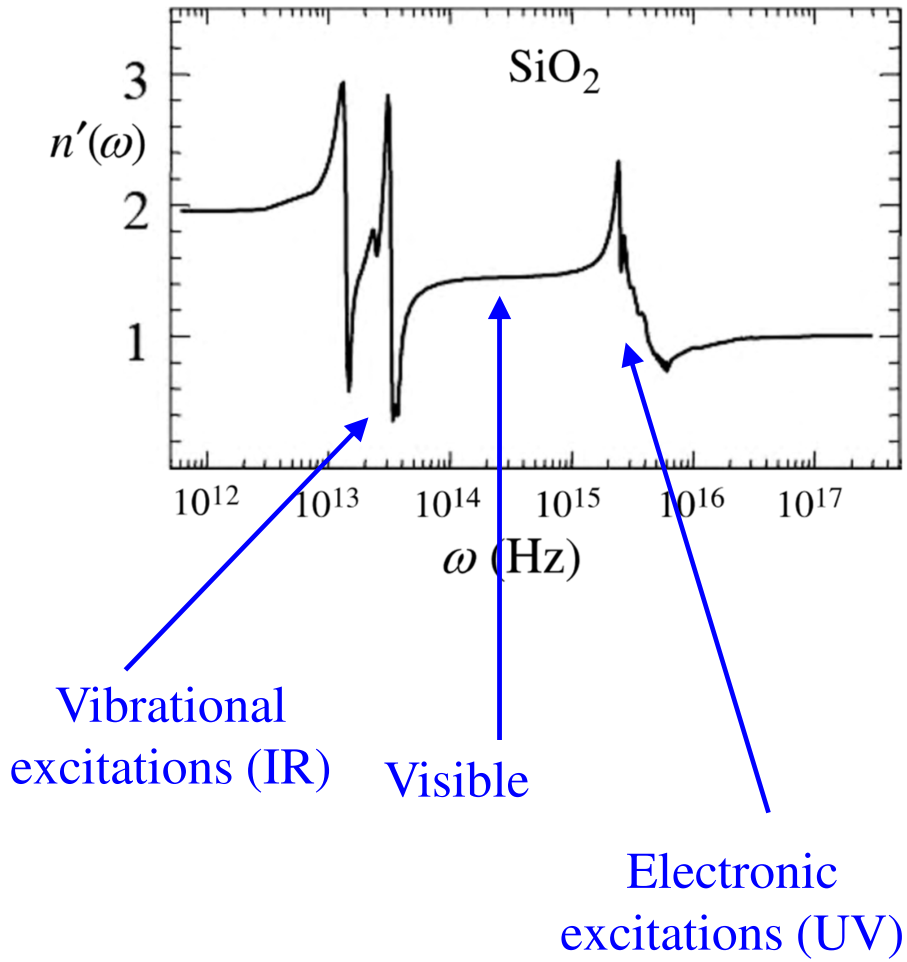


Wikipedia

Comparison of the Lorentz model with data

Silicon





Frequency dispersion and energy dissipation

$$\boxed{\int_V d^3r\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)=-\int_V d^3r\,\mathbf{j}_l\cdot\mathbf{E}-\int_V d^3r\,\nabla\cdot(\mathbf{E}\times\mathbf{H})}$$

Poynting theorem

+ complexes

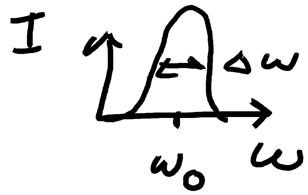
$$\boxed{\begin{aligned}\mathbf{D}(\mathbf{r},\omega)&=\hat{\varepsilon}(\omega)\mathbf{E}(\mathbf{r},\omega)\\ \mathbf{B}(\mathbf{r},\omega)&=\hat{\mu}(\omega)\mathbf{H}(\mathbf{r},\omega)\end{aligned}}$$

Lecture notes pp. 76-78
or Zangwill p. 627-629

$$\Downarrow$$

$$\boxed{\int_V d^3r\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)=\int_V d^3r\,\frac{\partial}{\partial t}u_{\mathrm{EM}}(t)+\int_V d^3r\,Q(t)}$$

Frequency dispersion and energy dissipation



$\frac{\Delta\omega}{\omega_0} \ll 1$

Energy dissipation occurs when the imaginary part of the permittivity is non-zero.

For c… e
frequ…

$$\begin{cases} u_{\mathrm{EM}}(t) = \dfrac{1}{2}\left\{ \dfrac{\partial}{\partial\omega}\left[\omega\varepsilon'(\omega)\right]\left\|\mathbf{E}(t)\right\|^2 + \dfrac{\partial}{\partial\omega}\left[\omega\mu'(\omega)\right]\left\|\mathbf{H}(t)\right\|^2 \right\} \\ Q(t) = \omega\left[\varepsilon''(\omega)\left\|\mathbf{E}(t)\right\|^2 + \mu''(\omega)\left\|\mathbf{H}(t)\right\|^2\right] \end{cases}$$

Is it possible to have $\varepsilon''(\omega) = 0$ when $\varepsilon' = \varepsilon'(\omega)$?

$u_{\mathrm{EM}}(t)$ is **the total energy per unit volume (transiently) stored in the medium**

$Q(t)$ is the **rate of energy absorption (per unit volume)**

↳ pertes

The consequences of causality:
Kramers-Kronig relations

χ : susceptibilité électrique

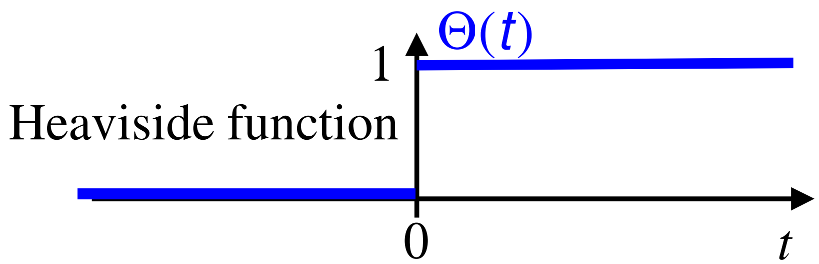
Recall: $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}; \quad \varepsilon = \varepsilon_0 \left(1+\chi\right)$

In order for causality to hold: $\mathbf{P}\left(\mathbf{r},t\right) = \varepsilon_0 \int_{-\infty}^{\infty} dt' \, \chi\left(t-t'\right) \mathbf{E}\left(\mathbf{r},t'\right)$

with $\boxed{\chi(\tau) = 0 \quad \text{for } \tau < 0}$

We can write $\boxed{\chi(t) = \Theta(t)\chi(t)}$

$\Big\downarrow$ TF

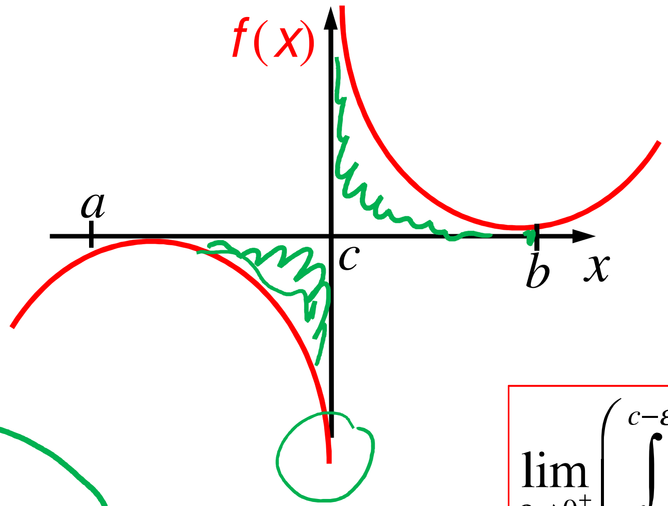


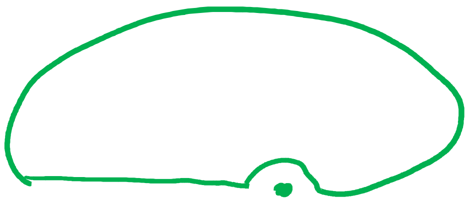
$$\hat{\chi}(\omega) = \frac{1}{2\pi} \hat{\Theta}(\omega) * \hat{\chi}(\omega)$$

$$\hat{\Theta}(\omega) = \pi\delta(\omega) + \text{p.v.}\frac{i}{\omega}$$

"valeur principale"

Cauchy principal value





$$\lim_{\varepsilon\to 0^+}\int_a^{c-\varepsilon} f(x)\,dx = -\infty$$

$$\lim_{\varepsilon\to 0^+}\int_{c+\varepsilon}^{b} f(x)\,dx = +\infty$$

$$\lim_{\varepsilon\to 0^+}\left(\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx\right) = \mathcal{P}\int_a^b f(x)\,dx$$

Cauchy principal value integral (if finite)

$$\text{p.v.}\frac{1}{x}: \quad \int_a^b \text{p.v.}\frac{1}{x}\,g(x)\,dx = \mathcal{P}\int_a^b \frac{g(x)}{x}\,dx$$

The consequences of causality: Kramers-Kronig relations

Recall: $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}; \quad \varepsilon = \varepsilon_0 (1+\chi)$

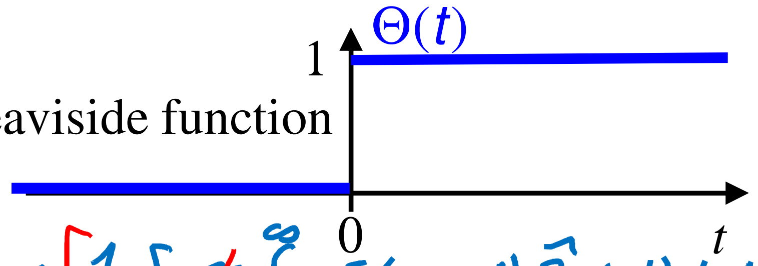
In order for causality to hold: $\mathbf{P}(\mathbf{r},t) = \varepsilon_0 \int_{-\infty}^{\infty} dt' \, \chi(t-t') \mathbf{E}(\mathbf{r},t')$

with $\boxed{\chi(\tau) = 0 \ \text{ for } \tau < 0}$

We can write $\boxed{\chi(t) = \Theta(t)\chi(t)}$

$$\hat{\chi}(\omega) = \frac{1}{2\pi}\hat{\Theta}(\omega) * \hat{\chi}(\omega)$$

$$\hat{\Theta}(\omega) = \pi\delta(\omega) + \text{p.v.}\frac{i}{\omega}$$



$$\hat{\chi}(\omega) = \left[\frac{1}{2\cancel{\pi}}\left[\cancel{\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega')\hat{\chi}(\omega')\,d\omega'\right]\right] = \frac{1}{2}\hat{\chi}(\omega)$$

$$+ \frac{1}{2\pi}\left[i\, P\int_{-\infty}^{\infty} \frac{\hat{\chi}(\omega')}{\omega-\omega'}\,d\omega'\right]$$

$$\hat{\chi}(\omega) = \frac{1}{\pi}\, i\, P\int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega-\omega'}\,d\omega'$$

The consequences of causality: Kramers-Kronig relations

Recall: $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$; $\varepsilon = \varepsilon_0 (1+\chi)$

$$\Rightarrow \quad \hat{\chi}(\omega) = \frac{i}{\pi} \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\hat{\chi}(\omega')}{\omega - \omega'}$$

$$\boxed{\begin{aligned} \chi'(\omega) &= -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\chi''(\omega')}{\omega - \omega'} \\ \chi''(\omega) &= \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\chi'(\omega')}{\omega - \omega'} \end{aligned}}$$

Kramers-Kronig relations

Link between real and imaginary parts

$\epsilon, \mu, \sigma \ldots$

Interpretation of the Kramers-Kronig relations

If $\varepsilon'(\omega)$ is not constant ***any***where, $\varepsilon''(\omega)$ is non-zero ***every***where

i.e., frequency dispersion in ***any*** interval of frequency implies that non-zero absorption occurs in ***every*** interval of frequency

Conversely, frequency dispersion occurs ***every***where in frequency if absorption occurs ***any***where in frequency.

Why is there this intimate relation between dispersion and energy dissipation?

$$\varepsilon = \varepsilon_0 (1+\chi)$$

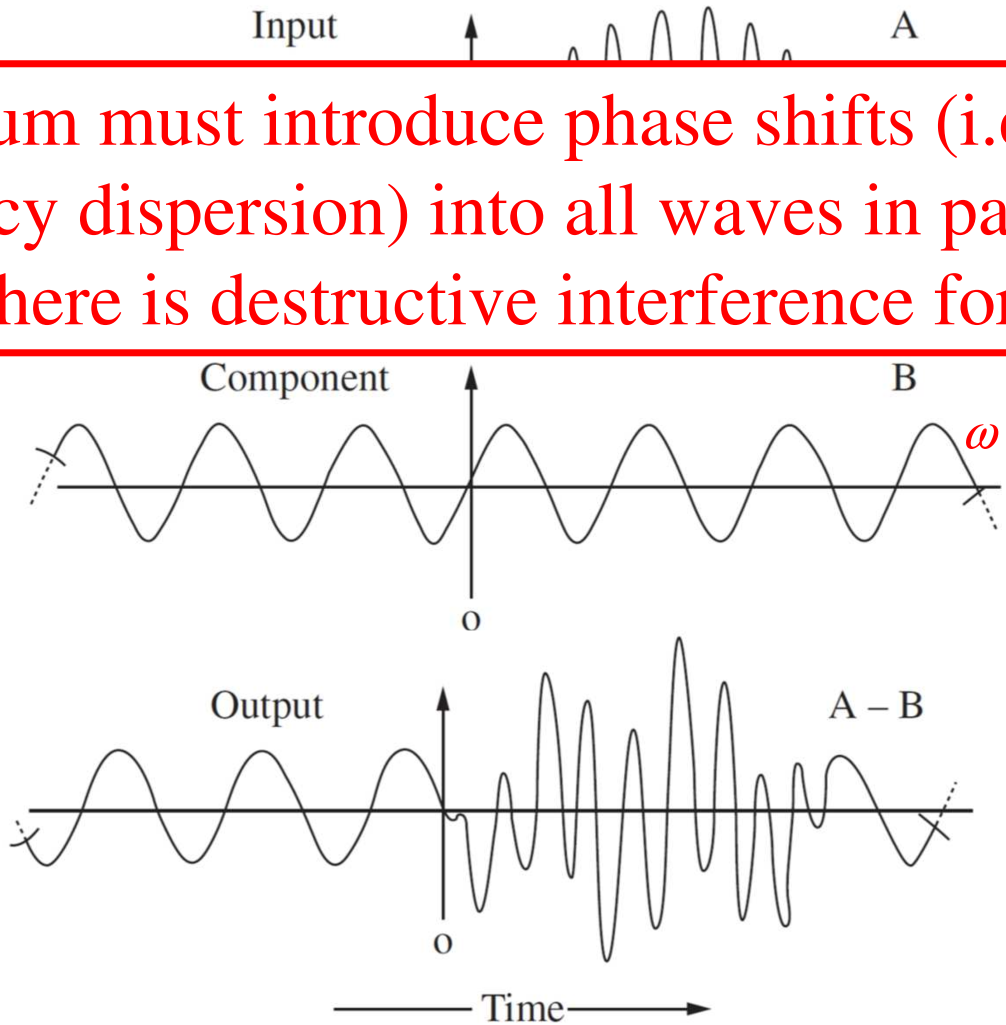
$$\chi'(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\chi''(\omega')}{\omega-\omega'}$$

$$\chi''(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\chi'(\omega')}{\omega-\omega'}$$

Frequency dispersion, absorption and causality

Wave packet

Medium must introduce phase shifts (i.e., frequency dispersion) into all waves in packet so that there is destructive interference for t<0



$\omega = \omega_0$

Assume material absorbs only for $\omega = \omega_0$

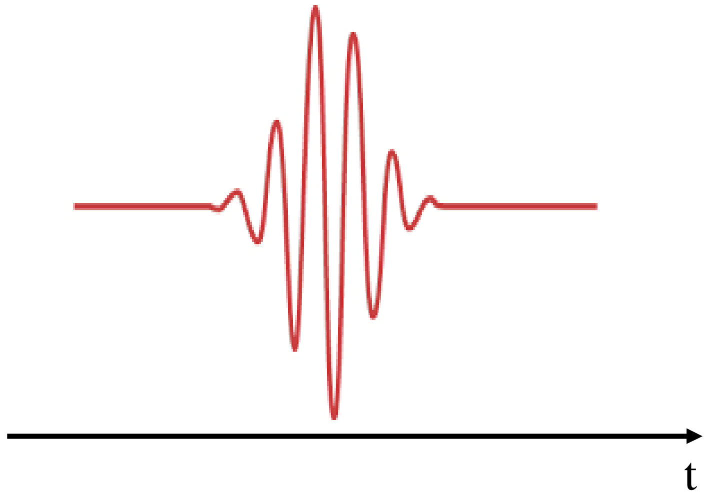
Result: signal before t=0 !!!

Zangwill, p.651

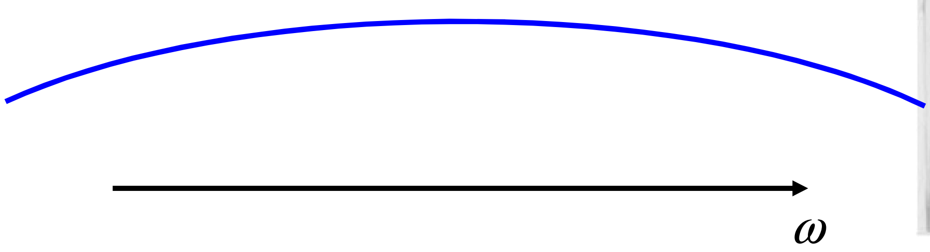
Paquet d'onde dans un milieu dispersif.

Short light pulses

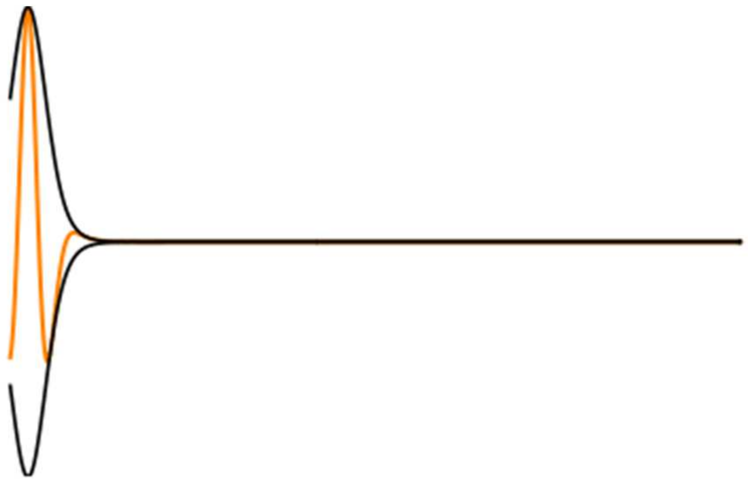
Pulse as a function of time



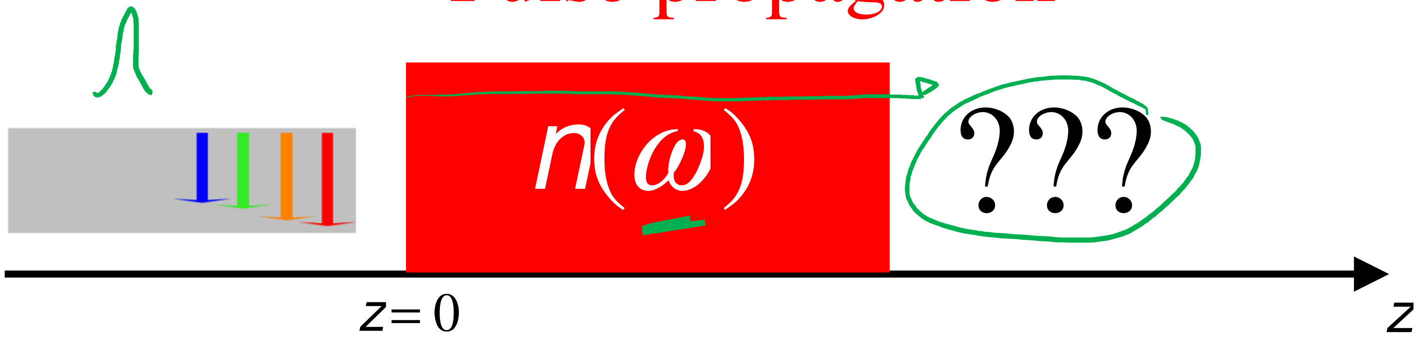
Frequency space



Consequence of dispersion



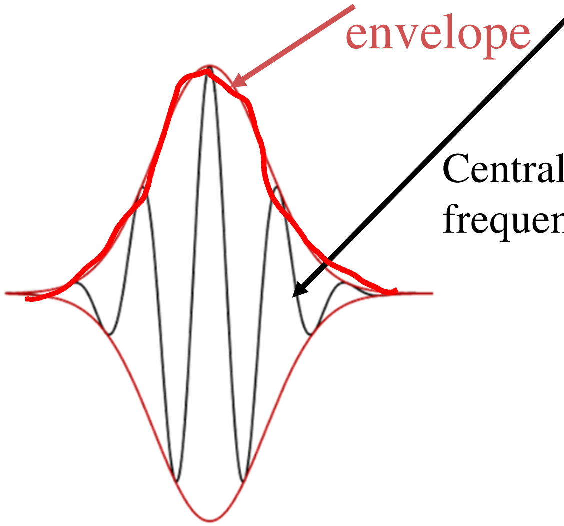
Pulse propagation



$$E(z=0,t) = \underbrace{\mathcal{E}(z=0,t)}\, e^{-i\omega_p t} + \text{c.c.} = E^{(+)}(z=0,t) + \text{c.c.}$$

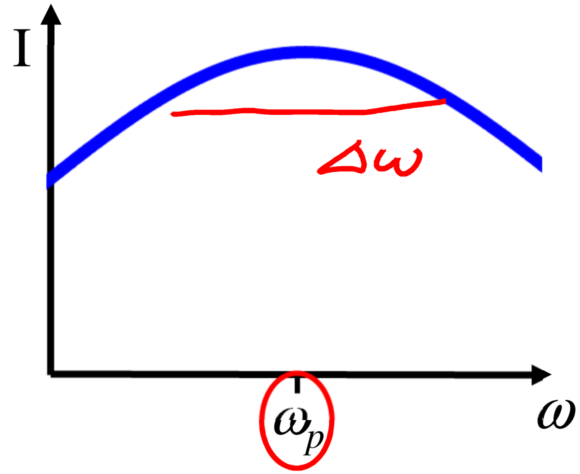
Analytic signal

ω_p "pulse" (PAS ~~Plasma~~)



Central frequency

Pulse as a function of time



Frequency space

$\frac{\Delta\omega}{\omega_p} \ll 1$

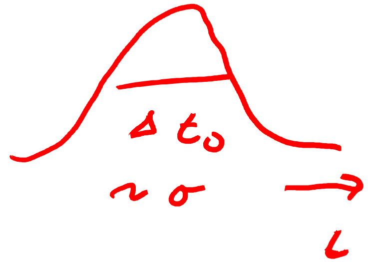
onde quasi-monochromatique

↳ impulsion "pas trop courte"

Pulse propagation

$$E(z=0,t)=\mathcal{E}(z=0,t)e^{-i\omega_p t}+\text{c.c.}=E^{(+)}(z=0,t)+\text{c.c.}$$

Example: Gaussian envelope $\mathcal{E}(z=0,t)=E_0\exp\left(-\frac{t^2}{2\Delta t_0^2}\right)$



$$\boxed{E^{(+)}(z=0,t)=E_0\exp\left(-\frac{t^2}{2\Delta t_0^2}\right)e^{-i\omega_p t}}$$

$\downarrow$ Fourier transform

Math review: Fourier transforms

$$F(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right) \longleftrightarrow F(\omega) = \sqrt{2\pi}\sigma \exp\left(-\frac{\omega^2\sigma^2}{2}\right)$$

$$e^{i\omega_0 t} F(t) \longleftrightarrow F(\omega + \omega_0)$$

Pulse propagation

$$E(z=0,t)=\underbrace{\mathcal{E}(z=0,t)}\,e^{-i\omega_p t}+\text{c.c.}=E^{(+)}(z=0,t)+\text{c.c.}$$

z=0

Example: Gaussian envelope $\mathcal{E}(z=0,t)=E_0\exp\left(-\frac{t^2}{2\Delta t_0^2}\right)$

$$\boxed{E^{(+)}(z=0,t)=E_0\exp\left(-\frac{t^2}{2\Delta t_0^2}\right)e^{-i\omega_p t}}$$

FT

TD

$$\boxed{E^{(+)}(z=0,\omega)=\hat{E}_0\exp\left(-\frac{(\omega-\omega_p)^2}{2\Delta\omega^2}\right)}$$

$$\hat{E}_0=\sqrt{2\pi}\Delta t_0 E_0$$

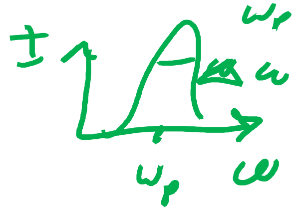
$$\boxed{\Delta\omega=1/\Delta t_0}$$

Pulse propagation

<u>*As we will see when we study diffraction*</u>, we can write an expression for the pulse at the entrance to the dispersive medium <u>***as a sum of monochromatic plane waves***</u>

$\frac{\Delta\omega}{\omega_p} \ll 1$

$$E^{(+)}(z=0,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}E^{(+)}(z=0,\omega)$$



Each component propagates with its own wave number $\boxed{k(\omega)=n(\omega)\frac{\omega}{c}}$

$$E^{(+)}(z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}E^{(+)}(z=0,\omega)\,e^{ik(\omega)z} \qquad \boxed{(4.73)}$$

Assume wave packet is sharply peaked at ω_p

➡ Taylor's expansion of $k(\omega)$ around ω_p (to second order)

Dispersion: $$\boxed{k(\omega)=n(\omega)\frac{\omega}{c}\simeq k(\omega_p)+(\omega-\omega_p)\left.\frac{dk}{d\omega}\right|_{\omega_p}+\frac{1}{2}(\omega-\omega_p)^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_p}}$$

Pulse propagation

$V_g = \frac{d\omega}{dk}\Big|_{\omega=\omega_p}$

Dispersion: $$\boxed{k(\omega) = n(\omega)\frac{\omega}{c} \simeq k(\omega_p) + (\omega-\omega_p)\underbrace{\frac{dk}{d\omega}\bigg|_{\omega_p}}_{\boxed{1/v_g}} + \frac{1}{2}(\omega-\omega_p)^2 \underbrace{\frac{d^2k}{d\omega^2}\bigg|_{\omega_p}}_{\boxed{\beta_2}}} \quad (4.75)$$

$$k(\omega) = n(\omega)\frac{\omega}{c}$$

$$\frac{dk}{d\omega} = \frac{dn}{d\omega}\frac{\omega}{c} + \frac{n}{c}$$

$$\boxed{v_g = \frac{c}{n + \omega\frac{dn}{d\omega}} = \frac{c}{n - \lambda\frac{dn}{d\lambda}}} \quad (4.76)$$

Group velocity

Group velocity dispersion

$$\boxed{\beta_2 = \frac{d^2k}{d\omega^2}\bigg|_{\omega_p}} \quad (4.77)$$

Substitute (4.75-4.77) in (4.73), evaluate integral!!!

$$E^{(+)}(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t} E^{(+)}(z=0,\omega)\, e^{ik(\omega)z} \quad (4.73)$$

TD

Propagation of a Gaussian pulse

$$E^{(+)}(z=0,\omega) = \hat{E}_0 \exp\left(-\frac{(\omega-\omega_p)^2}{2\Delta\omega^2}\right)$$

$$E^{(+)}(z=0,t) = E_0 \exp\left(-\frac{t^2}{2\Delta t_0^2}\right) e^{-i\omega_p t}$$

$$\beta_2 = \left.\frac{d^2k}{d\omega^2}\right|_{\omega_p}$$

z=0

Propagation in a dispersive medium

Phase velocity

$$v_\varphi = \frac{\omega_p}{k_p}$$

$$k_p \equiv k(\omega_p)$$

Envelope propagates at the group velocity v_g

$$E^{(+)}(z,t) \propto E_0 e^{-i(\omega_p t - k_p z)} \times \exp\left[-\frac{(t-z/v_g)^2}{2\Delta t(z)^2}\right] \exp\left[-i\frac{(t-z/v_g)^2}{2\Delta t(z)^2}\beta_2\Delta\omega^2 z\right]$$

$\beta_2 \neq 0$: pulse spreading

$\beta_2 \neq 0$: chirp

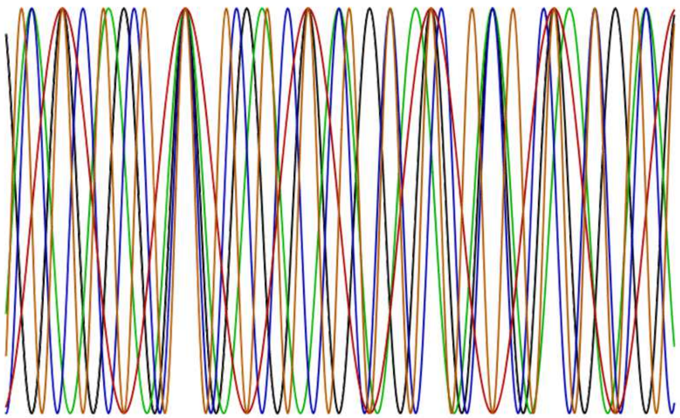
$$\Delta t(z)^2 = \Delta t_0^2 + \Delta\omega^2\beta_2^2 z^2$$

$$E^{(+)}(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t} E^{(+)}(z=0,\omega)\, e^{ik(\omega)z}$$

Phase and group velocities

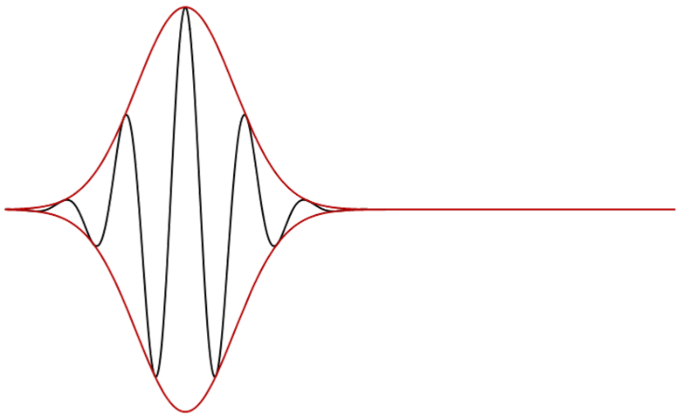
✔ Phase velocity:

$$\boxed{v_\varphi = \frac{\omega_p}{k_p} = \frac{c}{n(\omega_p)}}$$

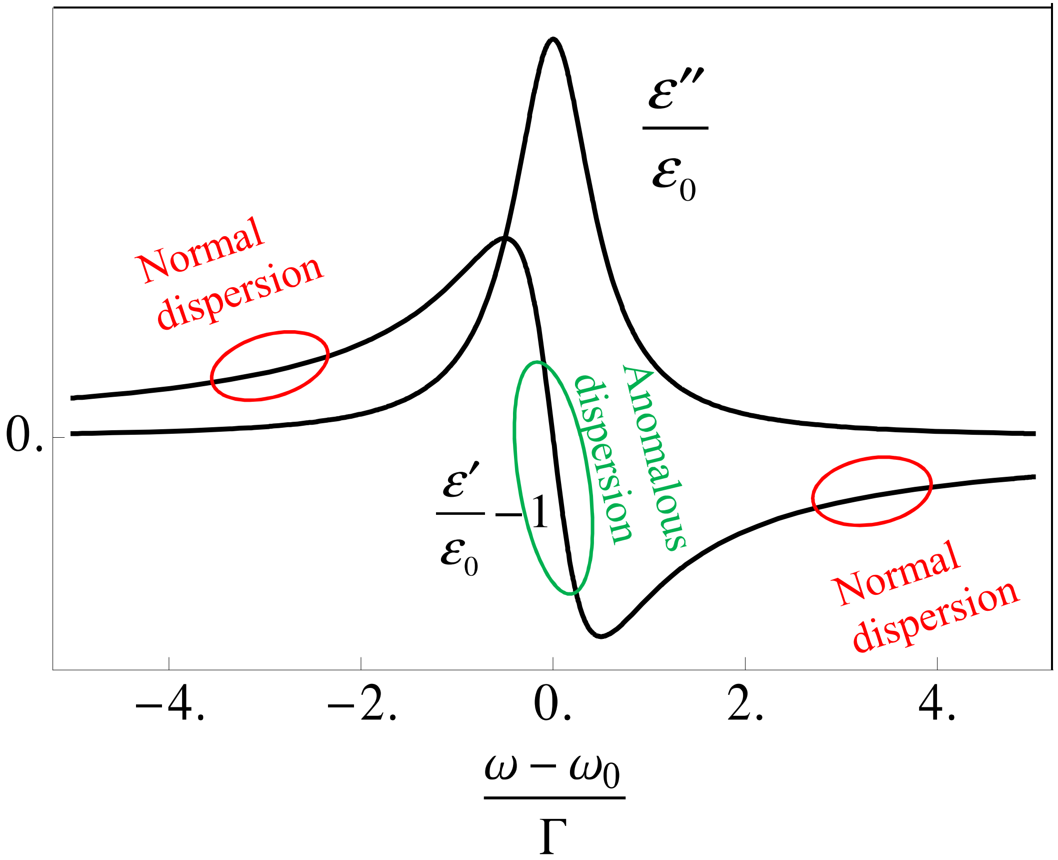


✔ Group velocity:

$$\boxed{v_g = \frac{c}{n + \omega \dfrac{dn}{d\omega}}}$$



Normal dispersion



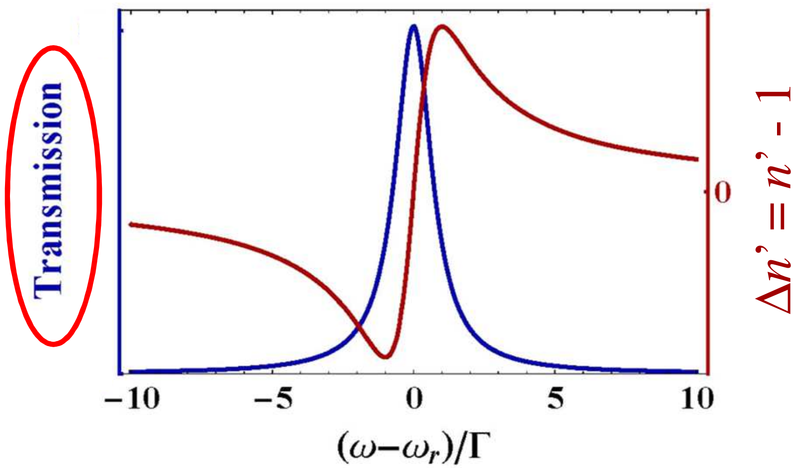
$$\boxed{\frac{dn}{d\omega} > 0}$$

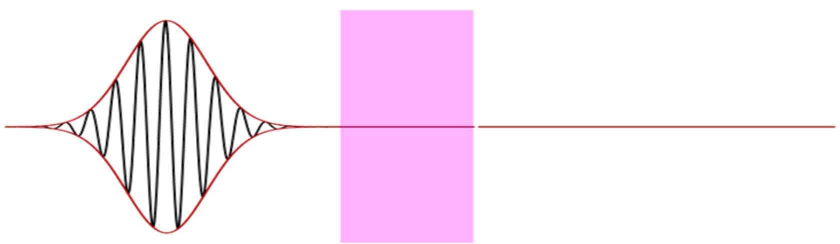
$$\boxed{v_g = \frac{c}{n + \omega \frac{dn}{d\omega}}}$$

Normal dispersion and slow light

$\frac{dn}{d\omega} >> 1$

Kramers-Kronig relations: link between dispersion and absorption





Peak in <u>*transmission*</u>:

↓

Normal dispersion--
positive slope

↓

Slow light

$$\boxed{v_g = \frac{c}{n + \omega \frac{dn}{d\omega}} \ll c}$$

Normal dispersion and slow light

Light speed reduction to 17 metres per second in an ultracold atomic gas

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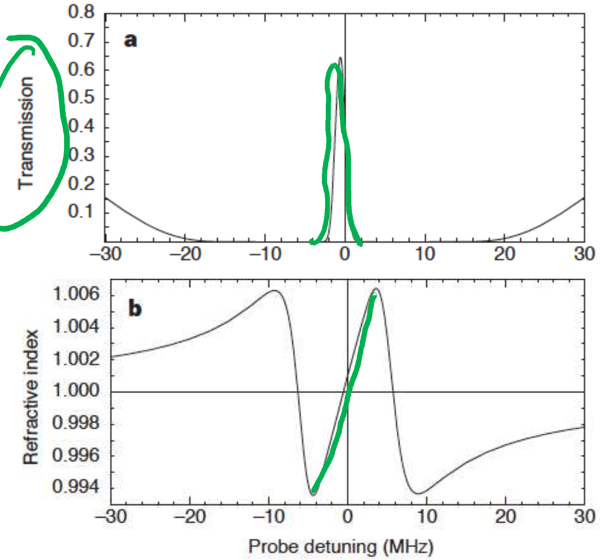
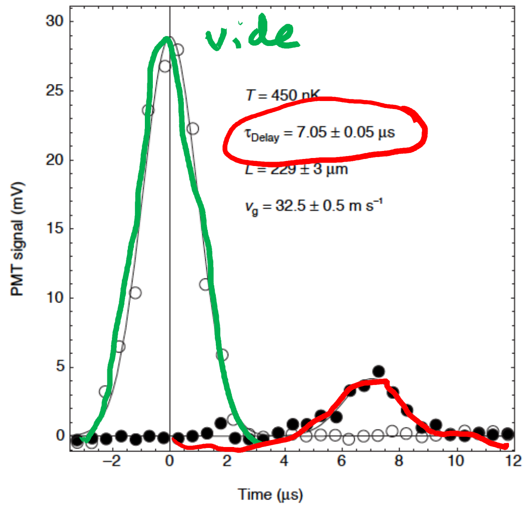
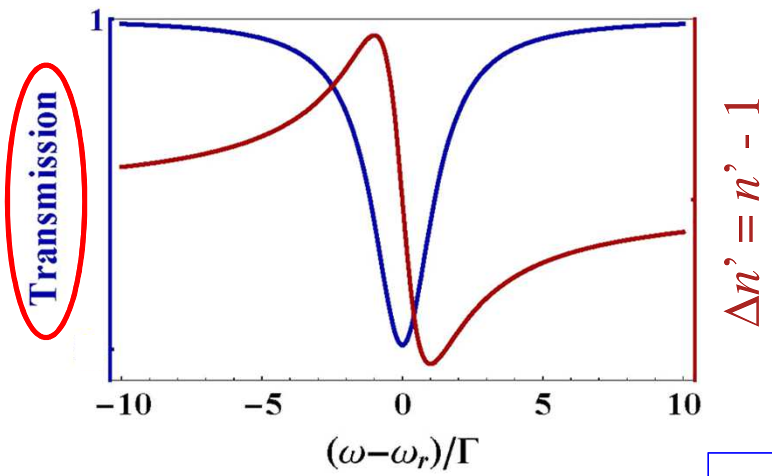




Anomolous dispersion and fast light

$\frac{dn}{d\omega} < 0$

Kramers-Kronig relations: link between dispersion and absorption



Peak in *absorption*:

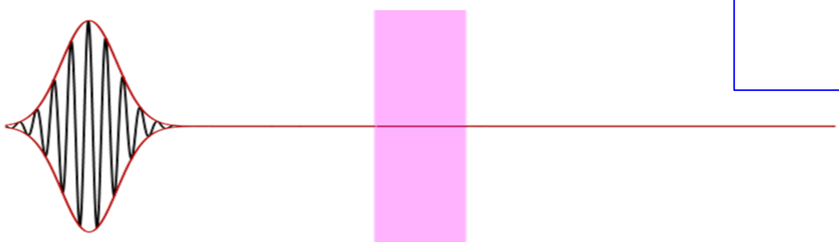
↓

Anomolous dispersion--
negative slope

↓

Fast light

$$v_g = \frac{c}{n + \omega \frac{dn}{d\omega}} > c \quad \text{or even } < 0$$



Anomolous dispersion and fast light ($v_g > c$)

The speed of information in a 'fast-light' optical medium

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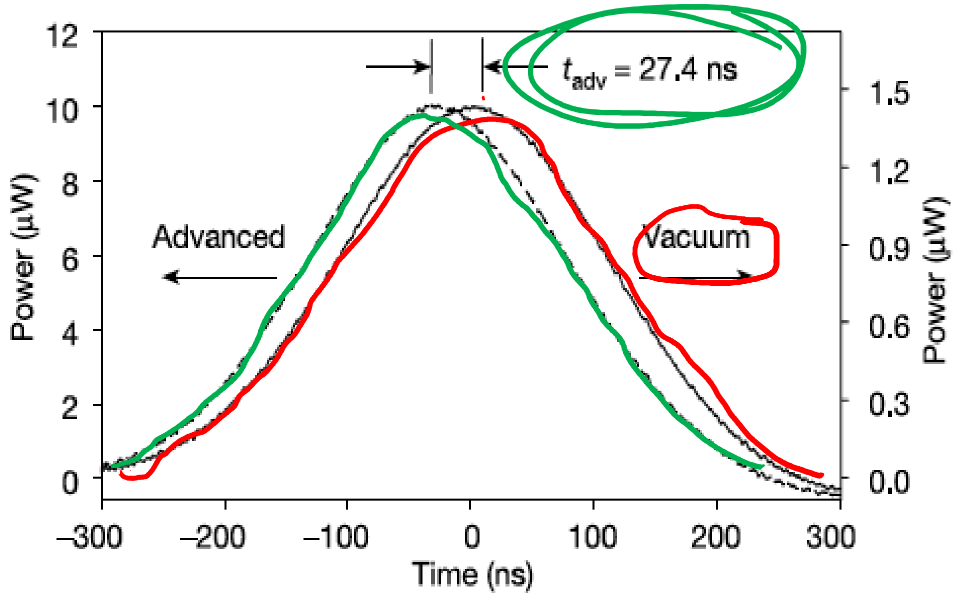
- What about causality?
- Can you transmit *information* faster than c?



information

1/0

Anomolous dispersion and fast light ($v_g > c$)

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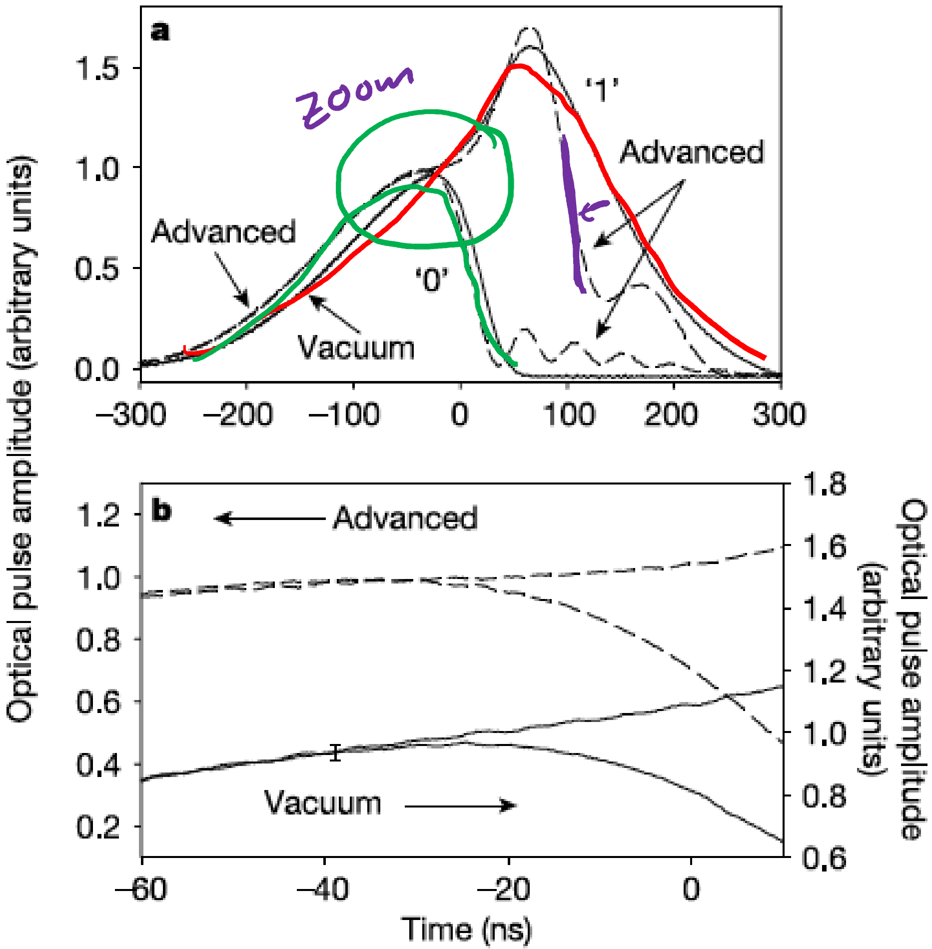
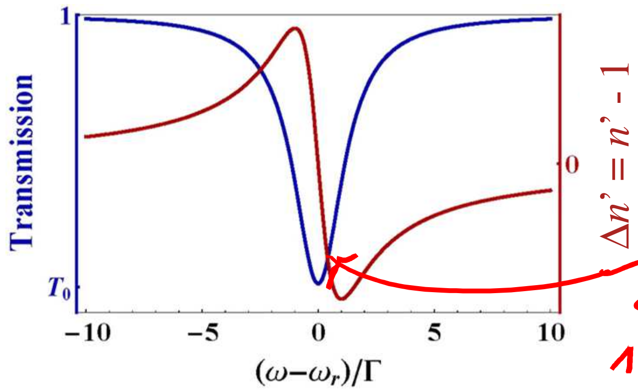
What about causality?



At what moment can you distinguish a 1 from a 0?



ici seulement 1 fréq "spécial"

A step edge contains a multitude of frequencies!

info = changement abrupte = plein de fréquence

Propagation of a Gaussian pulse

Phase velocity

$$v_\varphi = \frac{\omega_p}{k_p}$$

Envelop propagates at the group velocity v_g

$$E^{(+)}(z,t) \propto E_0 e^{-i(\omega_p t - k_p z)} \times \exp\left[-\frac{(t - z/v_g)^2}{2\Delta t(z)^2}\right] \exp\left[-i\frac{(t-z/v_g)^2}{2\Delta t(z)^2}\beta_2 \Delta\omega^2 z\right]$$

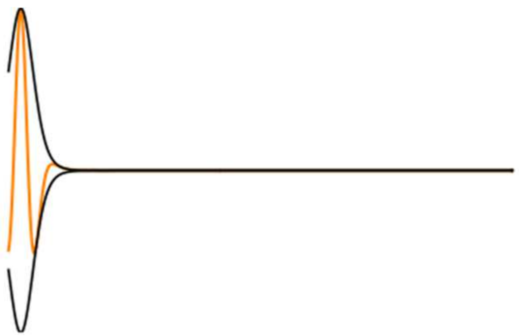
$\beta_2 \neq 0$: pulse spreading

$\beta_2 \neq 0$: chirp

$$\Delta t(z)^2 = \Delta t_0^2 + \Delta\omega^2 \beta_2^2 z^2$$

Consequences of $\beta_2 \neq 0$:

- Pulse spreads out in time
- Pulse becomes "chirped": instantaneous frequency varies linearly with time



Chirp: instantaneous frequency varies linearly with time

$$\boxed{E^{(+)}(z,t) \propto E_0 e^{-i(\omega_p t - k_p z)} \times \exp\left[-\frac{\left(t - z/v_g\right)^2}{2\Delta t(z)^2}\right] \exp\left[-i\frac{\left(t - z/v_g\right)^2}{2\Delta t(z)^2}\beta_2 \Delta\omega^2 z\right]}$$

$$E^{(+)}(t) \propto e^{-at^2} e^{-i\left(\omega_p t + bt^2\right)} \qquad \phi_{tot} = \omega_p t + bt^2$$

« chirp » → ω_i dép linéaire avec t

Instantaneous frequency: $\omega_i \equiv \dfrac{d\phi_{tot}}{dt} = \omega_p + 2bt$

$\boxed{\beta_2}$

Recall: $\boxed{k(\omega) = n(\omega)\dfrac{\omega}{c} \simeq k(\omega_p) + (\omega - \omega_p)\left.\dfrac{dk}{d\omega}\right|_{\omega_p} + \dfrac{1}{2}(\omega - \omega_p)^2 \left.\dfrac{d^2k}{d\omega^2}\right|_{\omega_p}}$

$k(\omega) = n(\omega)\dfrac{\omega}{c}$

$\sim \omega^2 \beta_2$

$\sim \beta_2 \omega \cdot \omega$

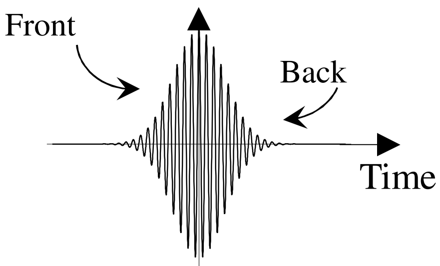
$\approx n(\omega)$

$v = \dfrac{c}{n(\omega)}$

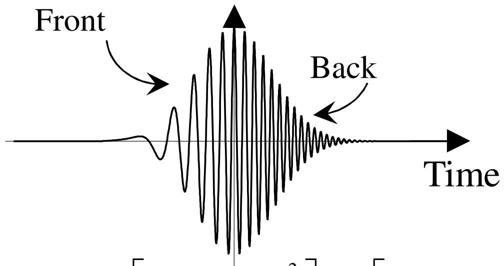
$\boxed{\beta_2 > 0\text{: high frequencies propagate more slowly than the low frequencies}}$

impulsion chirpée

Pulse spreading as a function of the initial pulse width

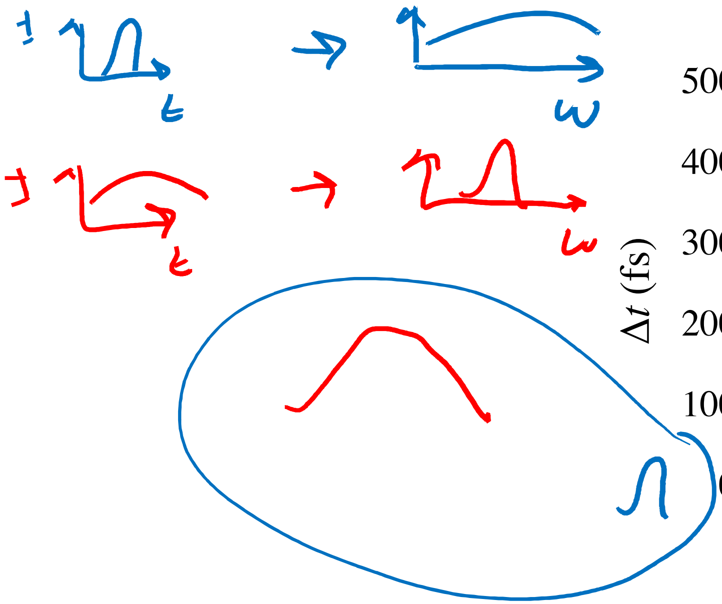


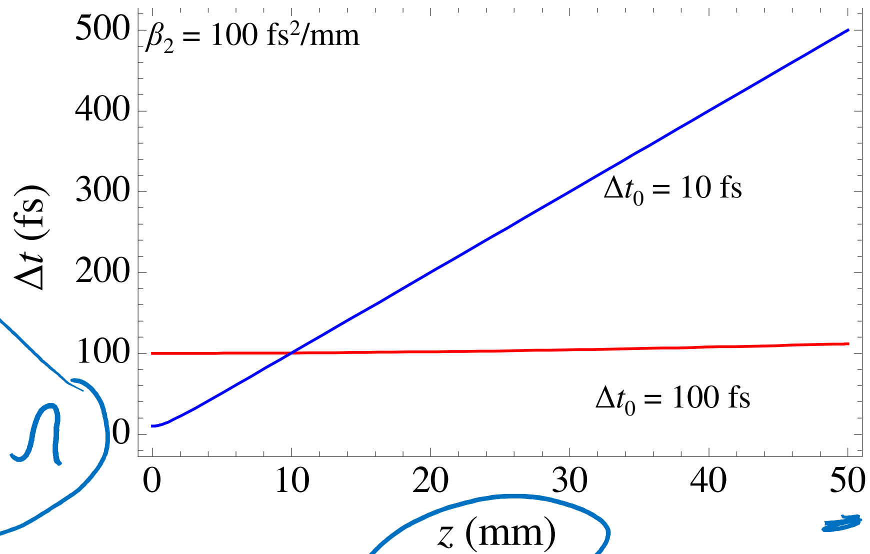
$n(\omega)$



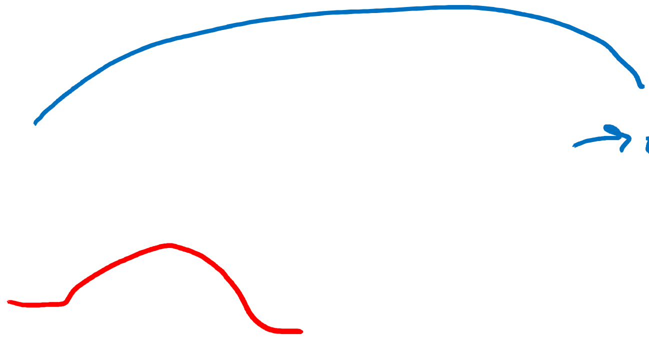
$$E^{(+)}(z=0,t) = E_0 \exp\left[-\frac{t^2}{\Delta t_0^2}\right] e^{-i\omega_p t}$$

$$E^{(+)}(z,t) \propto E_0 e^{-i(\omega_p t - k_p z)} \times \exp\left[-\frac{(t-z/v_g)^2}{2\Delta t(z)^2}\right] \exp\left[-i\frac{(t-z/v_g)^2}{2\Delta t(z)^2}\beta_2 \Delta\omega^2 z\right]$$





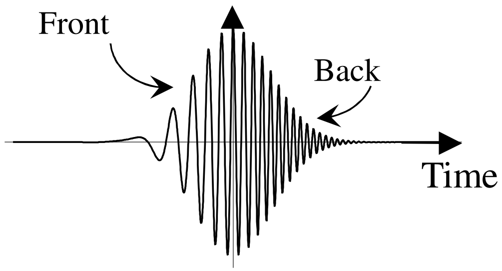
$$\boxed{\Delta t(z)^2 = \Delta t_0^2 + \Delta\omega^2 \beta_2^2 z^2}$$



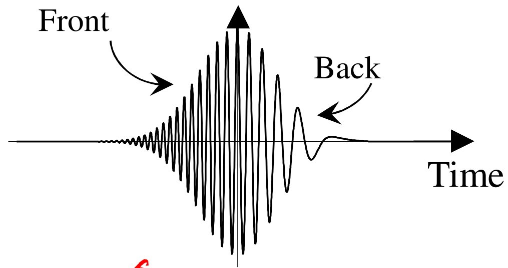
car plus de vitesses différentes présentes

The shorter the pulse, the larger the spectrum, the more it will spread!!!

Pulse spreading, pulse compression…

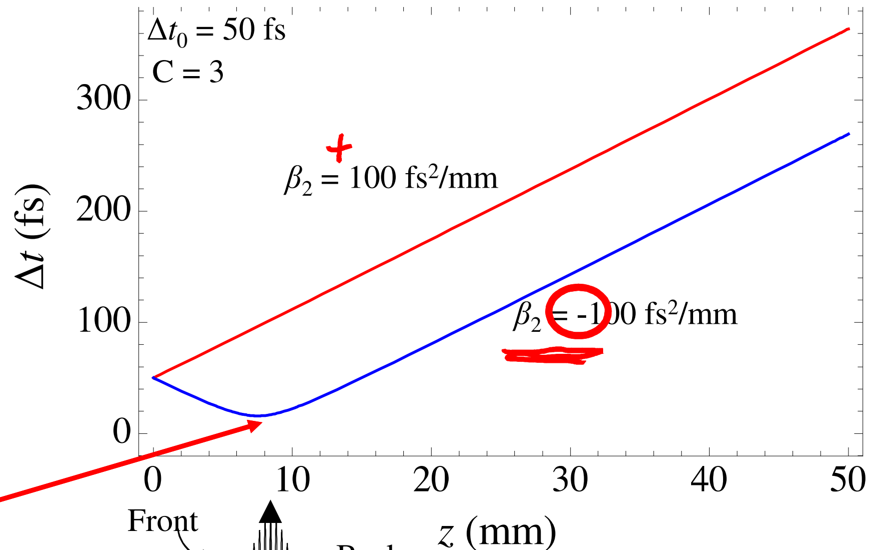


$n(\omega)$

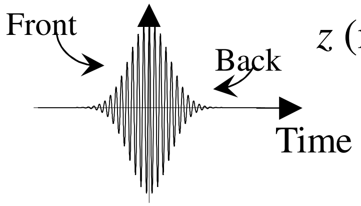


$$E^{(+)}(z=0,t) = E_0 \exp\left[-\frac{1+iC}{2}\frac{t^2}{\Delta t_0^2}\right] e^{-i\omega_p t}$$

Initial chirp



Pulse width decrease!



More in tutorial!

Summary

- Dispersion: optical response depends on frequency of light
- Origin of dispersion: the material cannot respond instantaneously!
- Consequences of dispersion:
  - Light pulse « shaping » (most often spreading…)
  - Energy dissipation in medium
- Speed(s) of light in matter: can have $0 < v_g < c$, $v_g > c$, $v_g < 0$.

Speed of ***signal*** is always in agreement with special relativity.

Reading

- Frequency dispersion:
  - Zangwill « Modern Electrodynamics », p. 624-629
- Kramers-Kronig relations
  - Zangwill, p. 649-653, Kittel « Introduction to Solid State Physics » p. 308-311 (7th edition)
- Lorentz model for dielectric matter
  - Zangwill, p. 635-636
- Wave packets in dispersive matter
  - Zangwill, p. 641-647
  - http://attolab.fr/
- Fast light
  - https://www.photonics.com/Articles/Fast_Light_Slow_Light_and_Optical_Precursors/a27833
- CPA
  - https://physicstoday.scitation.org/doi/pdf/10.1063/PT.3.4086