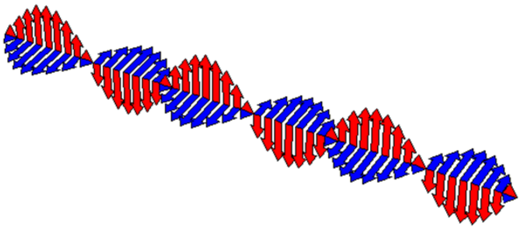
Wave Optics



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https://youtu.be/bZAs1W25_dQ
https://www.youtube.com/watch?v=nqqpkWicR2k

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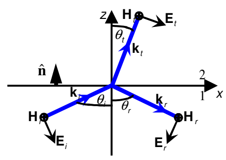
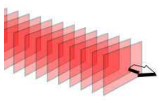
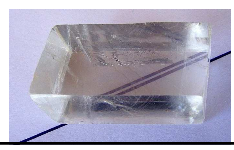
Course structure





- 8 x 2h lectures
- 8 x 2h tutorial sessions
- Final examination
- Lecture notes and recommended reading (eCampus)
- Ask questions!



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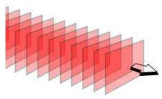
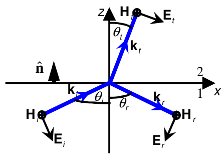
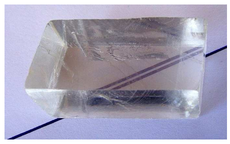
Goals of this course





- Plane waves: the Fresnel equations for reflection and refraction
- Matrix optics and optical cavities
- Propagation in dispersive media and causality

- Diffraction (plane and spherical waves)
- Wave guides
- Propagation in anisotropic media
- Polarization and wave plates
- Circular anisotropy



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Wave Optics

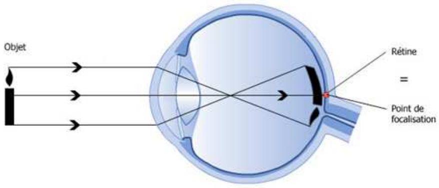
What is optics?

The study of…

*…light, its propagation, its production, the changes that it undergoes and produces, and other phenomena closely associated with it**

*Merriam Webster http://hdwpro.com/sunlight.html

Why study optics?



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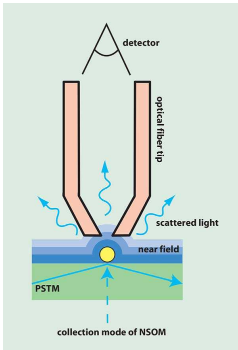
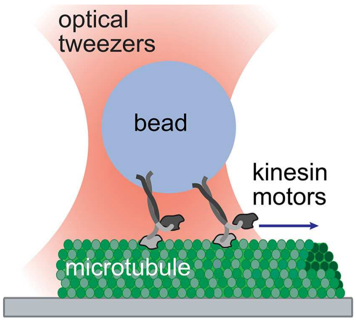
Why study optics?





Alain Aspect, Nobel Prize in Physics 2022

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Goals for today:

- Maxwell's equations and Poynting's theorem: energy conservation of EM fields in matter
- Plane waves
- Fresnel equations for reflection and refraction

If you wake up a physicist in the middle of the night and say "Maxwell" they are sure to say "electromagnetic field".
Rudolf Peierls (1962)

https://en.wikipedia.org/wiki/Rudolf_Peierls

Maxwell's equations in vacuum

In the presence of charges and currents

$$\nabla \times \mathbf{E} = \square \quad (1)$$

$$\nabla \times \mathbf{B} = \square\, \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \quad (2)$$

$$\nabla \cdot \mathbf{B} = \square \quad (3)$$

$$\nabla \cdot \mathbf{E} = \square \quad (4)$$

$\mathbf{j}$ = current per unit area perpendicular to flow

ρ = charge per unit volume

ε_0= vacuum permittivity

μ_0= vacuum permeability

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Maxwell's equations in matter

"Simple" matter:

- isotropic
- homogeneous
- non-dispersive
- non-magnetic
- linear dielectric

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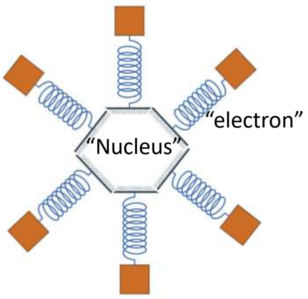
Dielectrics



Polarization **P**: dipole moment per unit volume

$$\rho_{bound} = \rho_b = -\nabla \cdot \mathbf{P}$$

$$\mathbf{j}_{polarization} = \frac{\partial \mathbf{P}}{\partial t}$$

ρ_{bound} = *bound* charge per unit volume

$\mathbf{j}_{polarization}$ = *polarization* current density = $\mathbf{j}_p$

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Linear dielectrics

In matter we define: $\mathbf{D} = (\varepsilon_0 \mathbf{E} + \mathbf{P})$

D: Electric displacement or auxiliary field.

For a *linear* material: $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$

χ_ε: Electric susceptibility

$$\mathbf{D} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon \mathbf{E}$$

ε

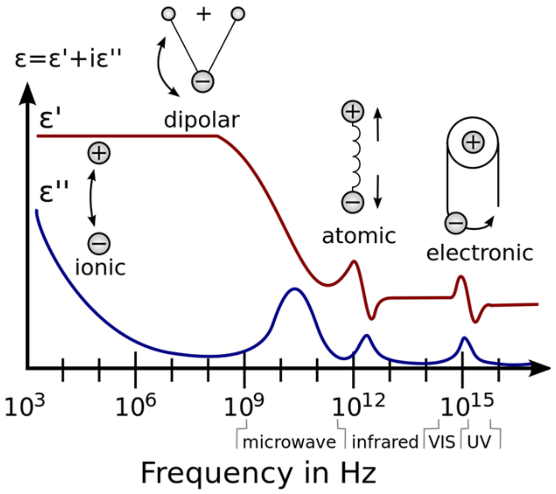
ε_0

ε_r

$\varepsilon_r = n^2$

Dielectrics

Permittivity as a function of frequency



Wikipedia

Maxwell's equations in matter:

For "simple" media

$$\mathbf{D} = \varepsilon \mathbf{E}$$
$$\mathbf{B} = \mu \mathbf{H}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (2.1)$$

$$\nabla \times \mathbf{H} = \mathbf{j}_{free} + \frac{\partial \mathbf{D}}{\partial t} \qquad (2.2)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (2.3)$$

$$\nabla \cdot \mathbf{D} = \rho_{free} \qquad (2.4)$$

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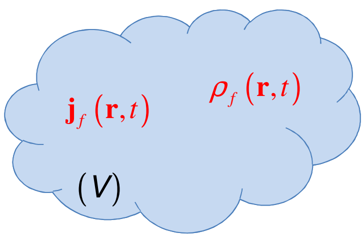
Poynting's theorem: conservation of energy for electromagnetic fields in matter

Goal: Find energy stored in EM fields and an expression for the energy flow

What is work done by EM forces on free charges in interval dt?

Recall: $\mathbf{j}(\vec{r},t) = \rho(\vec{r},t)\vec{\mathbf{v}}(\vec{r},t)$

Recall: $\mathbf{F}_{Lorentz} = q(\mathbf{E} + \vec{\mathbf{v}} \times \mathbf{B})$



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Poynting's theorem: conservation of energy for electromagnetic fields in matter

$$\frac{dW_{mech}}{dt} = \int_{vol} \vec{E} \cdot \vec{j}_f dV$$

$$\nabla \times \mathbf{H} = \mathbf{j}_{free} + \frac{\partial \mathbf{D}}{\partial t} \quad (2.2) \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2.1)$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})$$

$$\int_{vol} \mathbf{E} \bullet \mathbf{j}_f dV = \int_{vol} \left[\mathbf{H} \bullet \left(-\frac{\partial \mathbf{B}}{\partial t} \right) - \nabla \bullet (\mathbf{E} \times \mathbf{H}) - \mathbf{E} \bullet \frac{\partial \mathbf{D}}{\partial t} \right] dV$$

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Poynting's theorem: conservation of energy for electromagnetic fields in matter

$$\int_{vol} \left[\mathbf{E} \bullet \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \bullet \frac{\partial \mathbf{B}}{\partial t} \right] dV = -\int_{vol} \mathbf{E} \bullet \mathbf{j}_f dV - \int_{vol} \nabla \bullet (\mathbf{E} \times \mathbf{H}) dV$$

For linear media: $\mathbf{D} = \varepsilon \mathbf{E}$; $\mathbf{B} = \mu \mathbf{H}$ Note also: $\frac{1}{2} \varepsilon \frac{\partial}{\partial t} |\mathbf{E}|^2 = \varepsilon \mathbf{E} \bullet \frac{\partial \mathbf{E}}{\partial t}$

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Poynting's theorem: conservation of energy for electromagnetic fields in matter

$$\int_{vol}\left[\mathbf{E}\bullet\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\bullet\frac{\partial\mathbf{B}}{\partial t}\right]dV=-\int_{vol}\mathbf{E}\bullet\mathbf{j}_f\,dV-\int_{vol}\nabla\bullet(\mathbf{E}\times\mathbf{H})\,dV$$

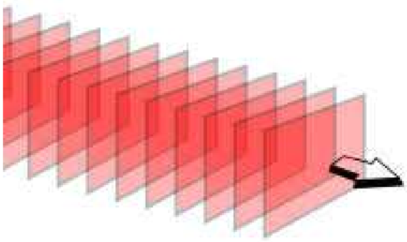
Poynting's theorem: conservation of energy for electromagnetic fields in matter

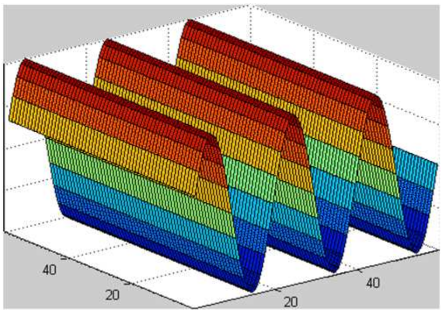
$$\int_{vol}\left[\mathbf{E}\bullet\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\bullet\frac{\partial\mathbf{B}}{\partial t}\right]dV=-\int_{vol}\mathbf{E}\bullet\mathbf{j}_f\,dV-\int_{vol}\nabla\bullet(\mathbf{E}\times\mathbf{H})\,dV$$

$$\boxed{\frac{dU_{EM}}{dt}=-\frac{dW_{mech}}{dt}-\int_S\mathbf{S}\cdot d\vec{\mathbf{a}}}$$

$$\boxed{\frac{\partial u_{EM}}{\partial t}=-\mathbf{j}_f\cdot\mathbf{E}-\nabla\cdot\mathbf{S}}$$

Plane waves





Plane waves: relations between **E,H,k**

Goal: find relations between **E, H** and **k**

"Simple" medium:

- homogeneous
- Isotropic
- non-dispersive
- (keep the possibility of magnetism)

- No free charges nor currents

Maxwell's equations in simple media with no free charges nor currents…

For "simple" media

$$\mathbf{D} = \varepsilon \mathbf{E}$$
$$\mathbf{B} = \mu \mathbf{H}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (2.1)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \qquad (2.19)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (2.3)$$

$$\nabla \cdot \mathbf{D} = 0 \qquad (2.21)$$

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…or more explicitly for linear media… spatially homogeneous

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad (2.26)$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \qquad (2.27)$$

$$\nabla \cdot \mathbf{H} = 0 \qquad (2.28)$$

$$\nabla \cdot \mathbf{E} = 0 \qquad (2.29)$$

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Intrinsic impedance of the medium, index of refraction

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\Omega$$

Recall: $$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

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Expression for a plane wave

- Fields are real quantities
- We use complex notation for convenience

Monochromatique plane wave:

$$\begin{cases} \mathbf{E}(\mathbf{r},t) = \mathcal{E}\exp\left[-i(\omega t - \mathbf{k}\cdot\mathbf{r})\right] + \text{c.c.} & (2.31) \\ \mathbf{H}(\mathbf{r},t) = \mathcal{H}\exp\left[-i(\omega t - \mathbf{k}\cdot\mathbf{r})\right] + \text{c.c.} & (2.32) \end{cases}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = \mathcal{E}\exp\left[-i(\omega t - \mathbf{k}\cdot\mathbf{r})\right] + \mathcal{E}^*\exp\left[+i(\omega t - \mathbf{k}\cdot\mathbf{r})\right]$$

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Relations between **E**, **H** and **k**

Substitute

$$\begin{cases}\mathbf{E}(\mathbf{r},t)=\mathcal{E}\exp\left[-i(\omega t-\mathbf{k}\cdot\mathbf{r})\right]+\text{c.c.}\\ \mathbf{H}(\mathbf{r},t)=\mathcal{H}\exp\left[-i(\omega t-\mathbf{k}\cdot\mathbf{r})\right]+\text{c.c.}\end{cases}$$

in $\nabla\times\mathbf{E}=-\mu\dfrac{\partial\mathbf{H}}{\partial t}$ (2.26)

Recall: $\nabla\times(f\mathbf{A})=f\nabla\times\mathbf{A}+\nabla f\times\mathbf{A}$

$$\mathbf{k}\times\mathcal{E}=\omega\mu\mathcal{H}$$ (2.34a)

Relations between **E**, **H** and **k**

Substitute

$$\begin{cases}\mathbf{E}(\mathbf{r},t)=\mathcal{E}\exp\left[-i(\omega t-\mathbf{k}\cdot\mathbf{r})\right]+\text{c.c.}\\ \mathbf{H}(\mathbf{r},t)=\mathcal{H}\exp\left[-i(\omega t-\mathbf{k}\cdot\mathbf{r})\right]+\text{c.c.}\end{cases}$$

in $\nabla\times\mathbf{H}=\varepsilon\dfrac{\partial\mathbf{E}}{\partial t}$ (2.27)

Recall: $\nabla\times(f\mathbf{A})=f\nabla\times\mathbf{A}+\nabla f\times\mathbf{A}$

$$\mathbf{k}\times\mathcal{H}=-\omega\varepsilon\mathcal{E}$$ (2.34b)

Relations between **E, H** and **k**

Substitute

$$\begin{cases}\mathbf{E}(\mathbf{r},t)=\mathcal{E}\exp\left[-i(\omega t-\mathbf{k}\cdot\mathbf{r})\right]+\text{c.c.}\\ \mathbf{H}(\mathbf{r},t)=\mathcal{H}\exp\left[-i(\omega t-\mathbf{k}\cdot\mathbf{r})\right]+\text{c.c.}\end{cases}$$

in

$$\nabla\cdot\mathbf{H}=0 \qquad (2.28)$$

$$\nabla\cdot\mathbf{E}=0 \qquad (2.29)$$

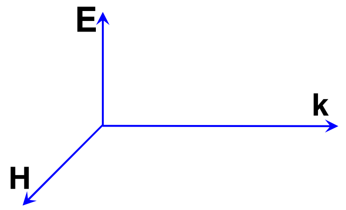
Recall: $\nabla\cdot(f\mathbf{A})=f\nabla\cdot\mathbf{A}+\nabla f\cdot\mathbf{A}$

$$\mathbf{k}\cdot\mathcal{H}=0 \quad \text{(2.33a)} \qquad\qquad \mathbf{k}\cdot\mathcal{E}=0 \quad \text{(2.33b)}$$

Relations between **E, H** and **k**

$$\mathbf{k}\times\mathcal{E}=\omega\mu\mathcal{H} \quad \text{(2.34a)}$$

$$\mathbf{k}\times\mathcal{H}=-\omega\varepsilon\mathcal{E} \quad \text{(2.34b)}$$



$$\mathbf{k}\cdot\mathcal{H}=0 \quad \text{(2.33a)}$$

$$\mathbf{k}\cdot\mathcal{E}=0 \quad \text{(2.33b)}$$

Ratio between **E** and **H**

$$\mathbf{k} \times \mathcal{E} = \omega\mu\mathcal{H} \quad (2.34a)$$

$$\mathbf{k} \times \mathcal{H} = -\omega\varepsilon\mathcal{E} \quad (2.34b)$$

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Dispersion relation

From $\mathbf{k} \times \mathcal{E} = \omega\mu\mathcal{H}$ (2.34a)

Recall: $v = \dfrac{1}{\sqrt{\mu\varepsilon}}$

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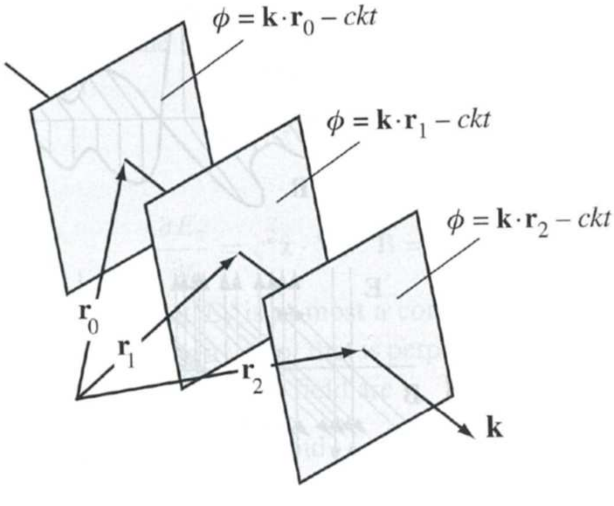
Phase velocity

$$\omega = \frac{ck}{n}$$

$$\mathbf{E}(\mathbf{r},t) = \mathcal{E}\exp\left[-i(\omega t - \mathbf{k}\cdot\mathbf{r})\right] + \mathcal{E}^{*}\exp\left[+i(\omega t - \mathbf{k}\cdot\mathbf{r})\right]$$

Phase velocity



At what speed does electromagnetic energy propagate?

Consider: $\langle \mathbf{S} \rangle$ time-averaged energy per unit surface per unit time

$\langle u_{\mathrm{EM}} \rangle$ time-averaged energy per unit volume

Unit analysis: $\dfrac{\langle \mathbf{S} \rangle}{\langle u_{\mathrm{EM}} \rangle}$

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Speed of energy propagation

Find $\langle \mathbf{S} \rangle$

$$= \left[\mathcal{E} \, \mathrm{e}^{i\phi} + \mathcal{E}^* \, \mathrm{e}^{-i\phi} \right] \times \left[\mathcal{H} \, \mathrm{e}^{i\phi} + \mathcal{H}^* \, \mathrm{e}^{-i\phi} \right] \quad \phi = \vec{k} \bullet \vec{r} - \omega t$$

Recall:

$$\left\langle e^{i2\phi} \right\rangle = 0$$

Eliminate $\mathcal{H}^*$ from $\mathcal{E} \times \mathcal{H}^*$

Recall: $\mathbf{k} \times \mathcal{E} = \omega \mu \mathcal{H}$ (2.34a)

Recall: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

➡ $\mathcal{E} \times \left(\mathbf{k} \times \mathcal{E}^* \right) =$

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Speed of energy propagation

Recall: $\frac{\omega}{k} = \frac{c}{n} = \frac{1}{\sqrt{\mu\varepsilon}}$ (2.37) $\qquad \mathcal{E} \times \mathcal{H}^* = \frac{1}{\omega\mu}\mathbf{k}\|\mathcal{E}\|^2$

What are we trying to do? Find $\frac{\langle \mathbf{S} \rangle}{\langle u_{\text{EM}} \rangle}$

Recall: $u_{EM} = \frac{1}{2}\left[\varepsilon\|\mathbf{E}\|^2 + \mu\|\mathbf{H}\|^2\right]$

$$= \left[\mathcal{E}\,\mathrm{e}^{i\phi} + \mathcal{E}^*\,\mathrm{e}^{-i\phi}\right] \bullet \left[\mathcal{E}\,\mathrm{e}^{i\phi} + \mathcal{E}^*\,\mathrm{e}^{-i\phi}\right]$$

Similarly $\left\langle \|\mathbf{H}\|^2 \right\rangle = 2\|\mathcal{H}\|^2$

Speed of energy propagation

$u_{EM} = \frac{1}{2}\left[\varepsilon\|\mathbf{E}\|^2 + \mu\|\mathbf{H}\|^2\right] \qquad \left\langle \|\mathbf{E}\|^2 \right\rangle = 2\|\mathcal{E}\|^2 \qquad \left\langle \|\mathbf{H}\|^2 \right\rangle = 2\|\mathcal{H}\|^2$

Recall: $\|\mathcal{H}\|^2 = \frac{\|\mathcal{E}\|^2}{Z^2}$

Thus: $\langle \mathbf{S} \rangle = \frac{2}{Z}\|\mathcal{E}\|^2\,\hat{k} \qquad \langle u_{EM} \rangle = 2\varepsilon\|\mathcal{E}\|^2\,\hat{k}$

Energy balance in simple matter but including absorption (no free charges nor currents => no mechanical work)

Let the electrical permittivity be complex! $\hat{\varepsilon} = \varepsilon' + i\varepsilon''$

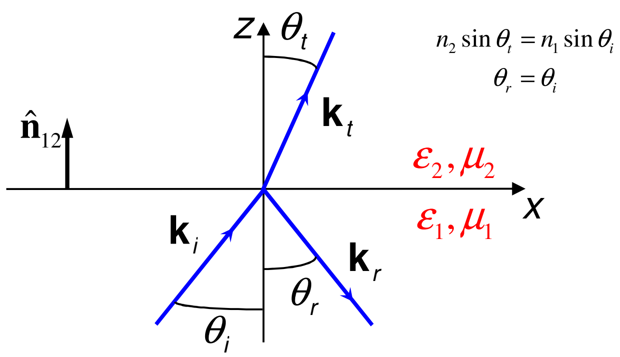
$$\mathcal{D} = \hat{\varepsilon}\mathcal{E} = \left(\varepsilon' + i\varepsilon''\right)\mathcal{E}$$

Amplitudes $\mathcal{E}(t)$ et $\mathcal{H}(t)$ are now decreasing functions of time!

$$\left\langle \oint_S dA\,\hat{\mathbf{n}}\cdot(\mathbf{E}\times\mathbf{H})\right\rangle + \frac{d}{dt}\int_V d^3r\left(\varepsilon'\|\mathcal{E}\|^2 + \mu\|\mathcal{H}\|^2\right) = -\int_V d^3r\; 2\omega\varepsilon''\|\mathcal{E}\|^2$$

Energy leaving volume per unit time | Change in stored energy per unit time | -Energy dissipated or absorbed in the medium

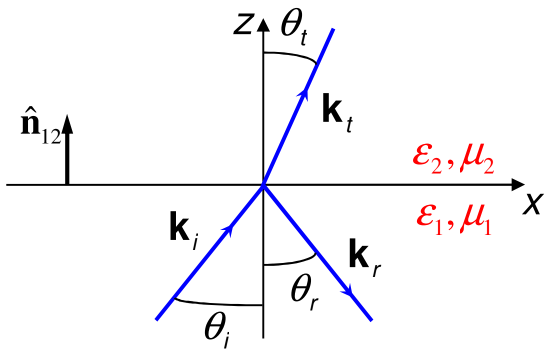
Reflection and refraction of plane waves at an interface



Goals: find Snell's equations
find Fresnel's equations

Incident, transmitted and reflected waves are all plane waves
Incident, transmitted and reflected waves have the same frequency

Reflection and refraction of plane waves at an interface



$$\mathbf{E}_1(\mathbf{r},t) = \left(\mathcal{E}_i \exp\left[-i\left(\omega t - \mathbf{k}_i \cdot \mathbf{r}\right)\right] + \mathcal{E}_r \exp\left[-i\left(\omega t - \mathbf{k}_r \cdot \mathbf{r}\right)\right] + \text{c.c.}\right) \quad (2.46)$$

$$\mathbf{E}_2(\mathbf{r},t) = \left(\mathcal{E}_t \exp\left[-i\left(\omega t - \mathbf{k}_t \cdot \mathbf{r}\right)\right] + \text{c.c.}\right) \quad (2.47)$$

Boundary conditions

$$\hat{\mathbf{n}}_{12} \cdot \left(\mathbf{D}_1 - \mathbf{D}_2\right) = 0 \quad (2.48)$$

$$\hat{\mathbf{n}}_{12} \cdot \left(\mathbf{B}_1 - \mathbf{B}_2\right) = 0 \quad (2.49)$$

⇨ The perpendicular components of D and B are continuous across the interface

$$\hat{\mathbf{n}}_{12} \times \left(\mathbf{E}_1 - \mathbf{E}_2\right) = 0 \quad (2.50)$$

$$\hat{\mathbf{n}}_{12} \times \left(\mathbf{H}_1 - \mathbf{H}_2\right) = 0 \quad (2.51)$$

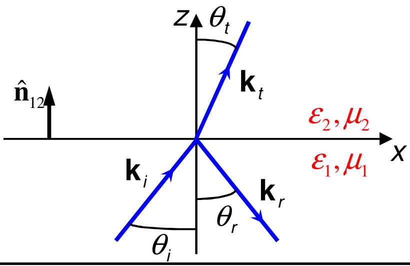
⇨ The tangential components of E and H are continuous across the interface

Snell's laws from boundary conditions

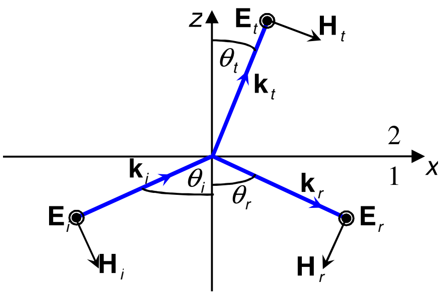
Use
$$\begin{cases} \mathbf{E}_1(\mathbf{r},t) = \left(\mathcal{E}_i \exp\left[-i\left(\omega t - \mathbf{k}_i \cdot \mathbf{r}\right)\right] + \mathcal{E}_r \exp\left[-i\left(\omega t - \mathbf{k}_r \cdot \mathbf{r}\right)\right] + \text{c.c.}\right) \\ \mathbf{E}_2(\mathbf{r},t) = \left(\mathcal{E}_t \exp\left[-i\left(\omega t - \mathbf{k}_t \cdot \mathbf{r}\right)\right] + \text{c.c.}\right) \end{cases}$$

and boundary conditions to find Snell's laws

The boundary conditions (2.48)-(2.51) must be satisfied for all instants in time, for all (x,y) on interface (i.e., z=0)



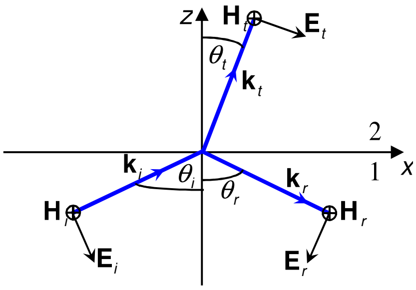
Fresnel equations

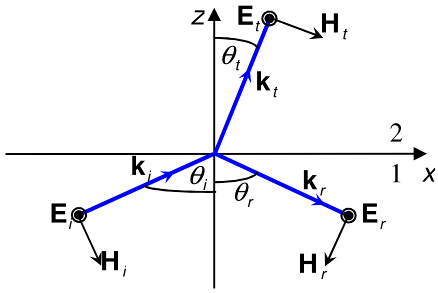


Find the reflection and transmission coefficients.

- How much of the wave is transmitted?
- How much of the wave is reflected?

Fresnel equations





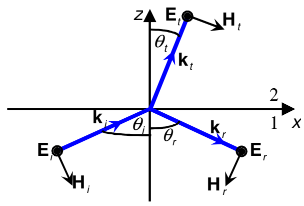
Fresnel equations: reflection and transmission coefficients

"s" polarization

$$r_s = \left[\frac{\mathcal{E}_r}{\mathcal{E}_i}\right]_s = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t} \qquad 2.66$$

$$t_s = \left[\frac{\mathcal{E}_t}{\mathcal{E}_i}\right]_s = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t} \qquad 2.67$$

"s" polarization

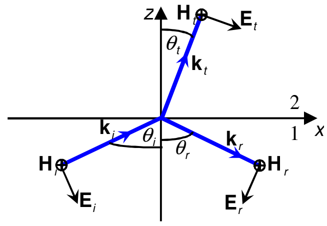


"p" polarization

$$r_p = \left[\frac{\mathcal{E}_r}{\mathcal{E}_i}\right]_p = \frac{Z_1\cos\theta_i - Z_2\cos\theta_t}{Z_1\cos\theta_i + Z_2\cos\theta_t} \qquad 2.61$$

$$t_p = \left[\frac{\mathcal{E}_t}{\mathcal{E}_i}\right]_p = \frac{2Z_2\cos\theta_i}{Z_1\cos\theta_i + Z_2\cos\theta_t} \qquad 2.62$$

"p" polarization



Fresnel equations: reflection and transmission coefficients

$$r_s = \left[\frac{\mathcal{E}_r}{\mathcal{E}_i}\right]_s = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}$$

$$t_s = \left[\frac{\mathcal{E}_t}{\mathcal{E}_i}\right]_s = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t}$$

$$Z_1 = \sqrt{\frac{\mu_0}{\varepsilon_1}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{n_1}$$

In terms of the indices of refraction for an interface between two dielectrics: $\mu_1 = \mu_2 = \mu_0$

"s" polarization

$$r_s = \left[\frac{\mathcal{E}_r}{\mathcal{E}_i}\right]_s = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t} \qquad 2.66$$

$$t_s = \left[\frac{\mathcal{E}_t}{\mathcal{E}_i}\right]_s = \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t} \qquad 2.67$$

Normal incidence

$$r_s = \left[\frac{\mathcal{E}_r}{\mathcal{E}_i}\right]_s = \frac{n_1 - n_2}{n_2 + n_1}$$

$$t_s = \left[\frac{\mathcal{E}_t}{\mathcal{E}_i}\right]_s = \frac{2n_1}{n_2 + n_1}$$

"p" polarization

$$r_p = \left[\frac{\mathcal{E}_r}{\mathcal{E}_i}\right]_p = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t} \qquad 2.61$$

$$t_p = \left[\frac{\mathcal{E}_t}{\mathcal{E}_i}\right]_p = \frac{2n_1\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t} \qquad 2.62$$

$$r_p = \left[\frac{\mathcal{E}_r}{\mathcal{E}_i}\right]_p = \frac{n_2 - n_1}{n_1 + n_2}$$

$$t_p = \left[\frac{\mathcal{E}_t}{\mathcal{E}_i}\right]_p = \frac{2n_1}{n_1 + n_2}$$

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Fresnel equations: air / perfect conductor interface

Normal incidence

$$Z_m = \sqrt{\frac{\mu_0}{\varepsilon_m}}$$

$$r_s = \left[\frac{\mathcal{E}_r}{\mathcal{E}_i}\right]_s = \frac{Z_2 - Z_1}{Z_2 + Z_1} \qquad r_p = \left[\frac{\mathcal{E}_r}{\mathcal{E}_i}\right]_p = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

$$t_s = \left[\frac{\mathcal{E}_t}{\mathcal{E}_i}\right]_s = \frac{2Z_2}{Z_2 + Z_1} \qquad t_p = \left[\frac{\mathcal{E}_t}{\mathcal{E}_i}\right]_p = \frac{2Z_2}{Z_1 + Z_2}$$

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"Impedance matching"

Normal incidence

$$r_s = \left[\frac{\mathcal{E}_r}{\mathcal{E}_i}\right]_s = \frac{Z_2 - Z_1}{Z_2 + Z_1} \qquad r_p = \left[\frac{\mathcal{E}_r}{\mathcal{E}_i}\right]_p = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

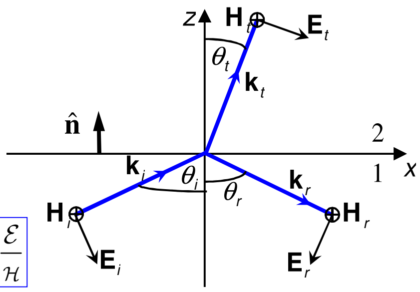
$$t_s = \left[\frac{\mathcal{E}_t}{\mathcal{E}_i}\right]_s = \frac{2Z_2}{Z_2 + Z_1} \qquad t_p = \left[\frac{\mathcal{E}_t}{\mathcal{E}_i}\right]_p = \frac{2Z_2}{Z_1 + Z_2}$$

Energy transport

How much energy (intensity) is **reflected**?

$$R = \frac{I_r}{I_i} = \left|\frac{\langle \mathbf{S}_r \rangle \cdot \hat{\mathbf{n}}}{\langle \mathbf{S}_i \rangle \cdot \hat{\mathbf{n}}}\right|$$

Recall: $\langle \mathbf{S} \rangle = \langle \mathbf{E} \times \mathbf{H} \rangle = \frac{2}{Z} \|\mathcal{E}\|^2 \hat{\mathbf{k}}$ $\quad Z = -\frac{\mathcal{E}}{\mathcal{H}}$



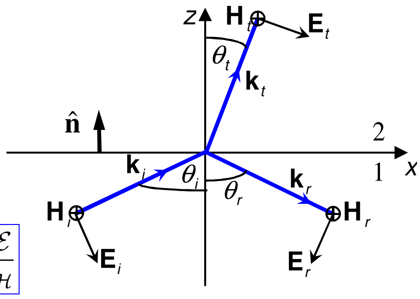
$$R = \left|\frac{\langle \mathbf{S}_r \rangle \cdot \hat{\mathbf{n}}}{\langle \mathbf{S}_i \rangle \cdot \hat{\mathbf{n}}}\right| = \left|\frac{\mathcal{E}_r}{\mathcal{E}_i}\right|^2 = |r|^2 \qquad (2.74)$$

Energy transport

How much energy (intensity) is **transmitted**?

$$T = \frac{I_t}{I_i} = \left|\frac{\langle \mathbf{S}_t \rangle \cdot \hat{\mathbf{n}}}{\langle \mathbf{S}_i \rangle \cdot \hat{\mathbf{n}}}\right|$$

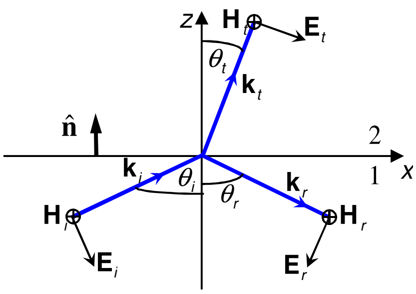
Recall: $\langle \mathbf{S} \rangle = \langle \mathbf{E} \times \mathbf{H} \rangle = \frac{2}{Z} |\mathcal{E}|^2 \, \hat{\mathbf{k}}$ $\qquad Z = -\frac{\mathcal{E}}{\mathcal{H}}$



$$T = \left|\frac{\langle \mathbf{S}_t \rangle \cdot \hat{\mathbf{n}}}{\langle \mathbf{S}_i \rangle \cdot \hat{\mathbf{n}}}\right| = \frac{Z_1 \cos\theta_t}{Z_2 \cos\theta_i} \left|\frac{\mathcal{E}_t}{\mathcal{E}_i}\right|^2 = \frac{Z_1 \cos\theta_t}{Z_2 \cos\theta_i} |t|^2 \quad (2.77)$$

Energy conservation

$$R + T = 1$$



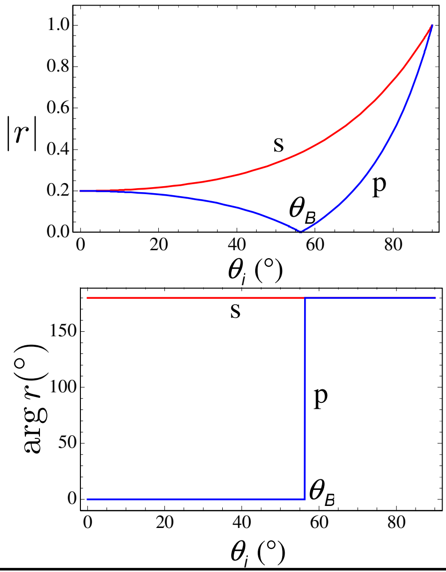
Reflection coefficient: amplitude and phase for an air/glass interface

$n_1 < n_2$

$\theta_B = \arctan \frac{n_2}{n_1}$

$r_s, r_p \in \mathrm{R}$

$\mu_1 = \mu_2 = \mu_0$



$n_1 = 1$

$n_2 = 1.5$

Reflection coefficient: amplitude and phase for a glass/air interface

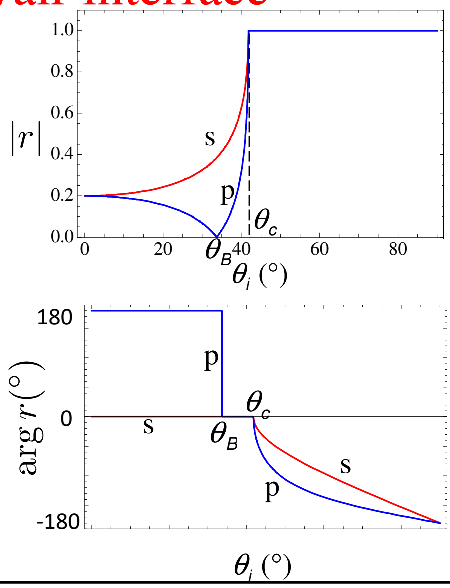
$n_1 > n_2$

$\theta_B = \arctan \frac{n_2}{n_1}$

$r_s, r_p \in \mathrm{C}$

$\theta_c = \arcsin \frac{n_2}{n_1}$

$\mu_1 = \mu_2 = \mu_0$

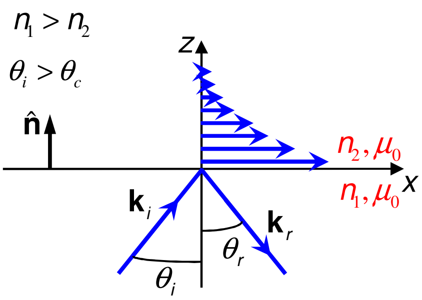


$n_1 = 1.5$

$n_2 = 1$

Total internal reflection: evanescent waves

$\mu_1 = \mu_2 = \mu_0$



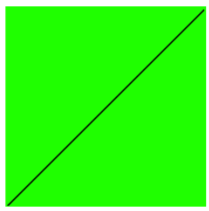
$$\mathbf{E}_t(\mathbf{r},t) = \mathcal{E}_t \exp\left[-i\left(\omega t - \mathbf{k}_t \cdot \mathbf{r}\right)\right] + \text{c.c.}$$
$$= \mathcal{E}_t \exp\left[-i\left(\omega t - k_{tx}x - k_{tz}z\right)\right] + \text{c.c.}$$

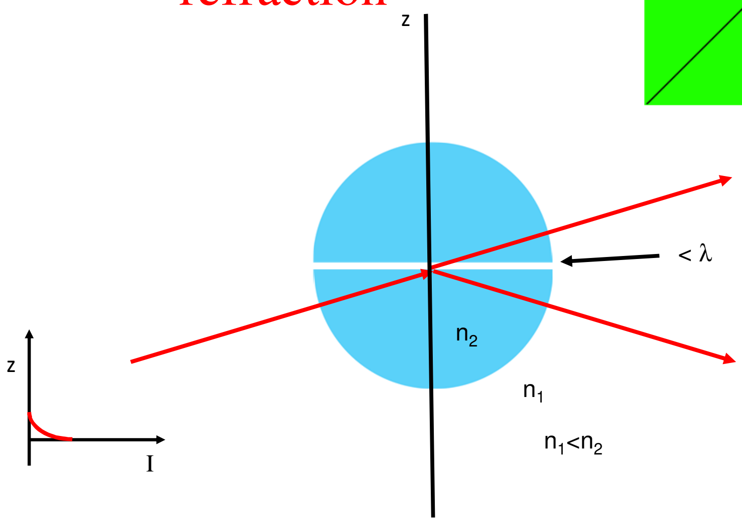
$$k_{tz} = i\kappa$$

$$\mathbf{E}_t(\mathbf{r},t) = \mathcal{E}_t \exp(-\kappa z) \exp\left[-i\left(\omega t - k_{tx}x\right)\right] + \text{c.c.}$$

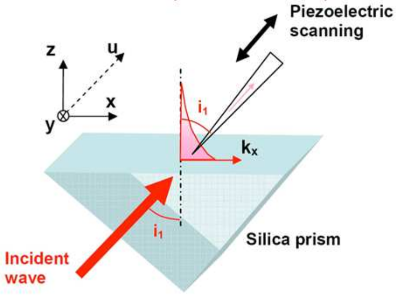
Evanescent wave: propagates in the x-direction, decays exponentially in the z-direction

Frustrated total internal refraction

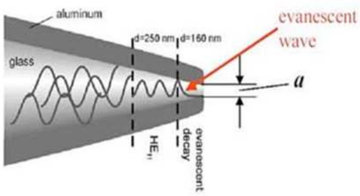


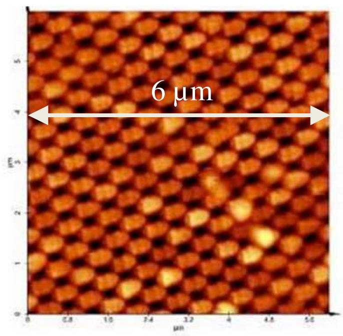


Scanning Near-Field Optical Microscope (SNOM): Effet tunnel optique



http://iramis.cea.fr





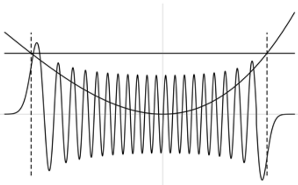
This is a topographic picture showing a Scanning Near-Field Optical Microscopy (SNOM) image of a sub-micrometric triangular pattern of holes drilled on polymethyl methacrylate (PMMA) by electron beam lithography and wet etching, performed in the Materials and Microsystems Laboratory.

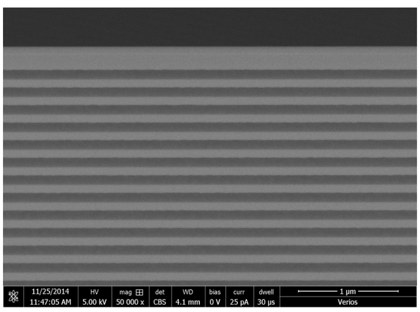
http://www.azonano.com

Lecture 2: Fresnel's equations and applications, geometrical optics

Goals today

- Applications of Fresnel's equations: Fabry-Perot etalon, Bragg mirror
- Matrix Optics (geometrical optics)
- Relation between geometrical and wave optics: the eikonal equation





Refraction Matrix

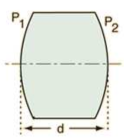
$$\begin{bmatrix} 1 & P \\ 0 & 1 \end{bmatrix}$$

P = surface power or lens power

Translation Matrix

$$\begin{bmatrix} 1 & 0 \\ -\frac{d}{n} & 1 \end{bmatrix}$$

d = lens thickness or lens separation



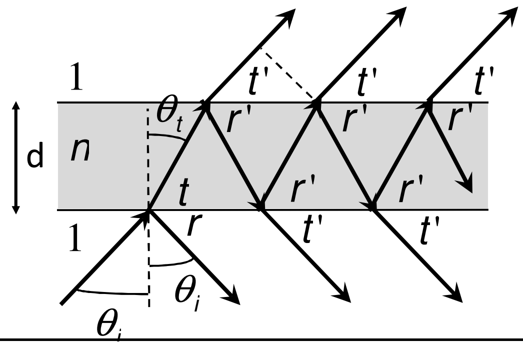
$$\begin{bmatrix} 1 & P_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{d}{n} & 1 \end{bmatrix}\begin{bmatrix} 1 & P_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - P_2\frac{d}{n} & P_1 + P_2 - P_1P_2\frac{d}{n} \\ -\frac{d}{n} & 1 - P_1\frac{d}{n} \end{bmatrix}$$

System Matrix

Fabry-Perot interferometer

t, r : transmission and reflection coefficients, **$n_1 < n_2$**

t', r' : transmission and reflection coefficients, **$n_1 > n_2$**



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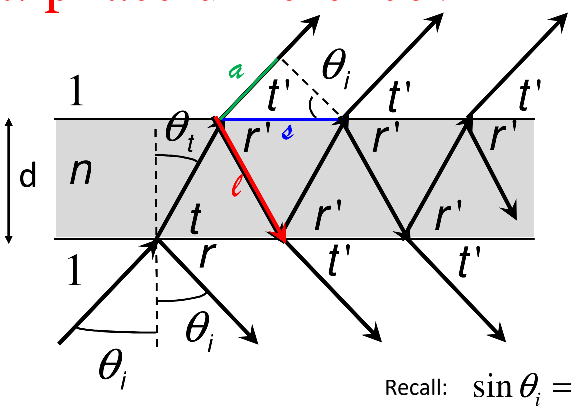
Fabry-Perot: phase difference?



Recall: $\sin\theta_i = n\sin\theta_t$

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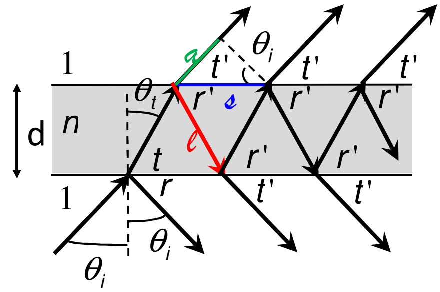
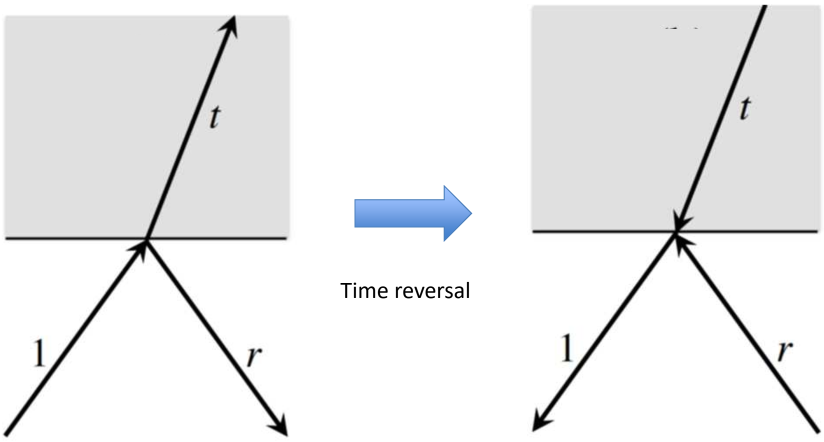
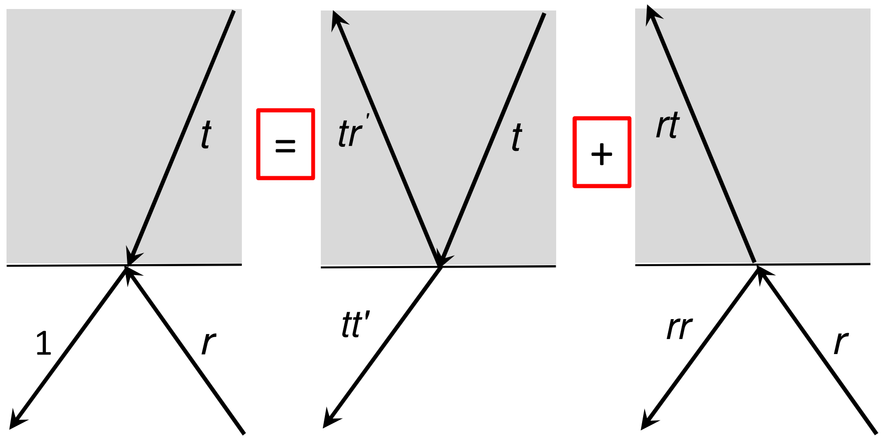
Fabry-Perot interferometer



Stokes relations



Stokes relations



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Fabry Perot interferometer

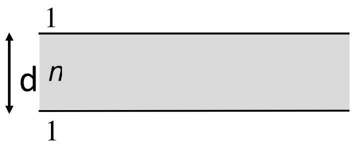
$$\mathcal{E}_t = \mathcal{E}_i \frac{tt'}{1 - r'^2 e^{i\Delta\phi}} \qquad (2.92)$$

$$\begin{aligned} &r^2 + tt' = 1 \\ &tr' + rt = 0 \\ &r = -r' \\ &tt' = 1 - r^2 = 1 - R \end{aligned} \qquad (2.93)\text{-}(2.96)$$

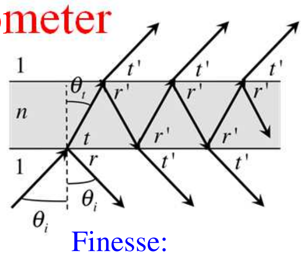
$$\frac{I_t}{I_i} = \left|\frac{\mathcal{E}_t}{\mathcal{E}_i}\right|^2 = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2\frac{\Delta\phi}{2}}$$

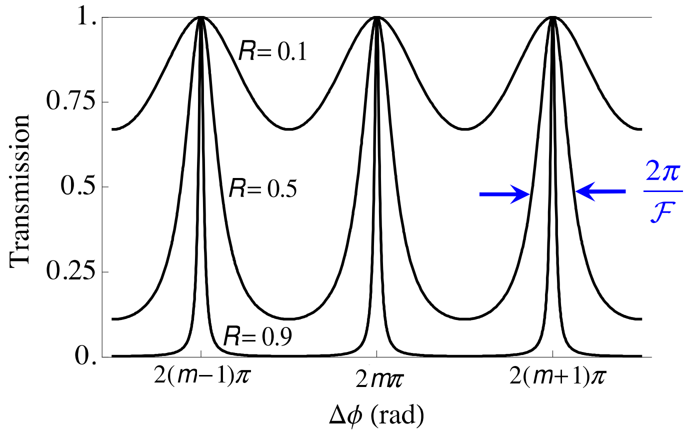
Resonance?

$$\Delta\phi = 2n\frac{\omega}{c}d\cos\theta_t$$



Fabry-Perot interferometer





Finesse:

$$\mathcal{F} = \frac{\pi\sqrt{R}}{1-R}$$

$$R = \left(\frac{n-1}{n+1}\right)^2$$
$$= 0.25 \text{ for } n = 3$$

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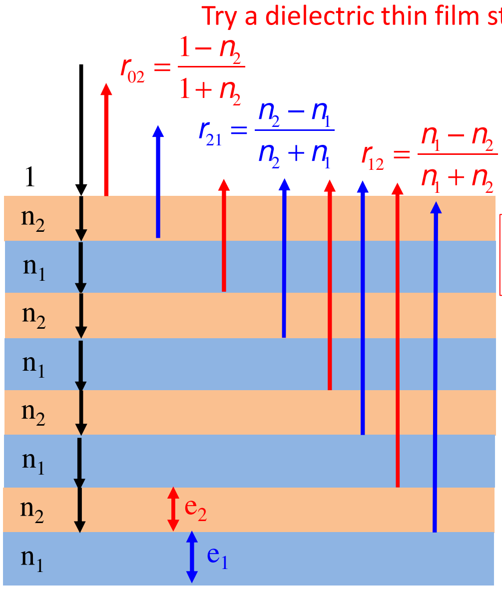
Highly reflective mirror

Metallic mirror: R ~ 99 %. Can we do better?

Try a dielectric thin film stack!



$r_{21} = -r_{12}$

Need another π phase change for each round trip in a layer

$$\frac{2\pi}{\lambda} n_1 2e_1 = \frac{2\pi}{\lambda} n_2 2e_2 = \pi$$

Layer thickness = $\frac{\lambda}{4n}$

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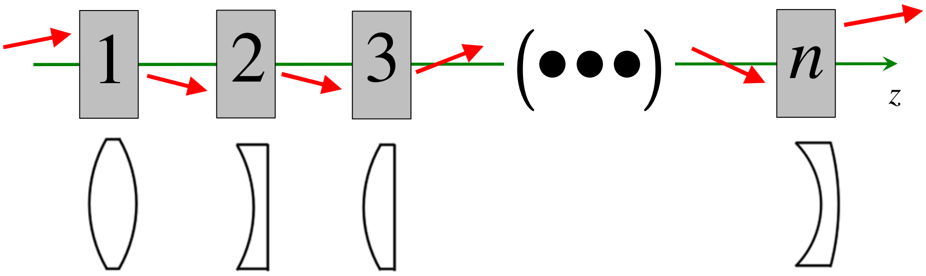
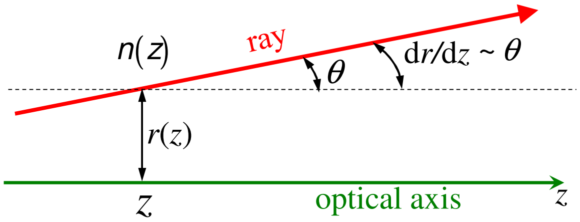
Geometrical vs wave optics

Matrix optics



$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathbf{M}_n \mathbf{M}_{n-1} \dots \mathbf{M}_2 \mathbf{M}_1$$

Matrix optics (for paraxial rays)



$n(z)$: local index of refraction

Reduced slope

$$r'(z) \equiv n(z)\frac{\mathrm{d}r}{\mathrm{d}z}$$

Ray vector: $\mathbf{R}(z) \equiv \begin{pmatrix} r(z) \\ r'(z) \end{pmatrix}$

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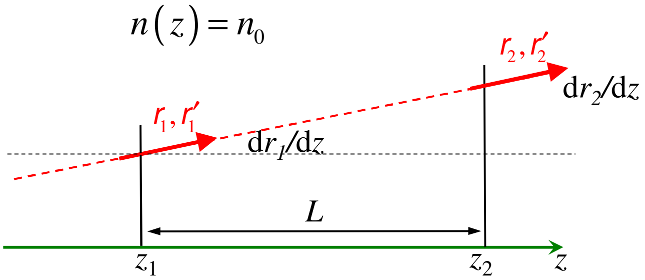
Matrix optics: free space propagation

Reduced slope

$$r'(z) \equiv n(z)\frac{\mathrm{d}r}{\mathrm{d}z}$$

$$n(z) = n_0$$



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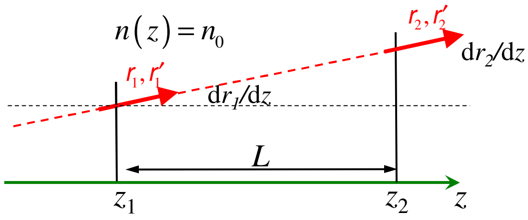
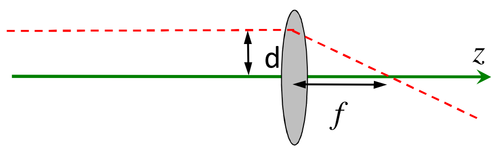
Matrix optics: free space propagation

$$r_2 = r_1 + \frac{r_1'}{n_0} L$$

$$r_1' = r_2'$$



Matrix optics: thin lens of focal length *f* in air



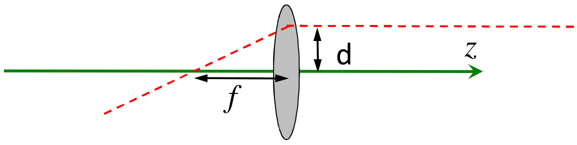
Just before the lens:

Just after the lens:

Matrix optics: thin lens of focal length *f* in air

A = 1 ; C = - 1/*f*

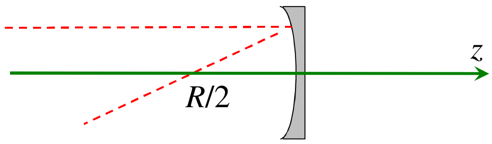
Just before the lens:



Just after the lens:

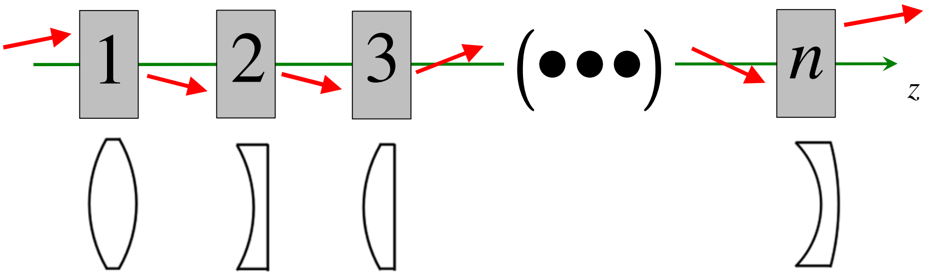
$f > 0$ for a convergent lens

Matrix optics: sphercial mirror of focal length *f*

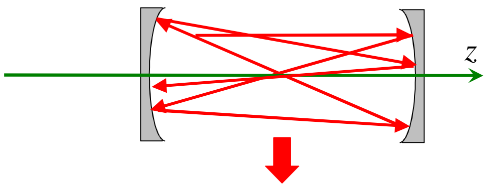


$R > 0$ for a concave mirror

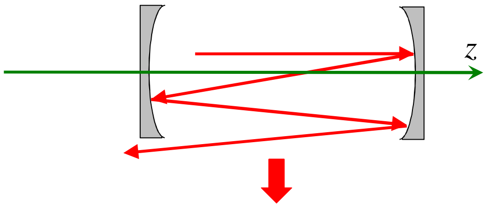
Matrix optics and an optical system



Matrix optics and cavity stabiliy



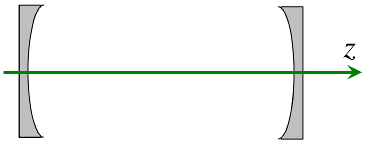
STABLE CAVITY!



UNSTABLE CAVITY!

Cavity as a series of periodic elements

M_{mirror_L} M_{prop} M_{mirror_R}



=

M_{prop}	M_{mirror_R}	M_{prop}	M_{mirror_L}	M_{prop}	M_{mirror_R}	etc...

1 period = 1 round trip ⇨ $\mathbf{M}_{RT}$

Plan of attack for investigating cavity stability

1. Find eigenvalues of (general) round-trip matrix $\mathbf{M}_{RT}$ ⇨ $\lambda_{\pm}$
2. Find corresponding eigenvectors ⇨ $\mathbf{r}_{+}$, $\mathbf{r}_{-}$
3. Express initial ray $\mathbf{r}_{0}$ in terms of these eigenvectors
4. Find an expression for $\mathbf{r}_{n}$ in terms of these eigenvectors and eigenvalues and examine it.

Find eigenvalues

1. Find eigenvalues of (general) round-trip matrix $\mathbf{M}_{RT}$ ➡ $\lambda_\pm$

$$\lambda_\pm = m \pm \sqrt{m^2 - 1} \text{ with } m = \frac{A+D}{2}$$

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Cavity stability: steps 2 to 4

2. Eigenvectors: $\mathbf{M}_{RT}\mathbf{r}_+ = \lambda_+\mathbf{r}_+$, $\mathbf{M}_{RT}\mathbf{r}_- = \lambda_-\mathbf{r}_-$

3. Let $\mathbf{r}_0 = a\mathbf{r}_+ + b\mathbf{r}_-$

4. $\mathbf{r}_n$ = ?

Condition for stability?

Recall: $\lambda_\pm = m \pm \sqrt{m^2 - 1} \text{ with } m = \frac{A+D}{2}$ ➡ $-1 \le m \le 1$

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Cavity stability

$$\lambda_\pm = m \pm \sqrt{m^2 - 1} \text{ with } m = \frac{A+D}{2} \quad \Rightarrow -1 \le m \le 1$$

Let $\frac{A+D}{2} = \cos\theta$

$$\Rightarrow \quad \lambda_\pm = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

Recall: $\mathbf{r}_n = \mathbf{M}_{RT}^n \mathbf{r}_0 = \mathbf{M}_{RT}^n (a\mathbf{r}_+ + b\mathbf{r}_-) = a\lambda_+^n \mathbf{r}_+ + b\lambda_-^n \mathbf{r}_-$

$$\mathbf{r}_n = ae^{in\theta}\mathbf{r}_+ + be^{-in\theta}\mathbf{r}_- = \mathbf{r}_0 \cos n\theta + \mathbf{s}_0 \sin n\theta$$

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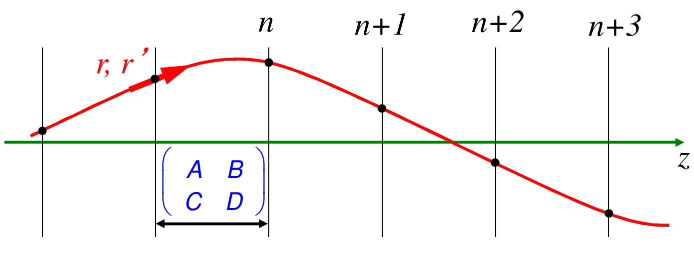
Stable cavity



$$\mathbf{r}_n = ae^{in\theta}\mathbf{r}_+ + be^{-in\theta}\mathbf{r}_- = \mathbf{r}_0 \cos n\theta + \mathbf{s}_0 \sin n\theta$$

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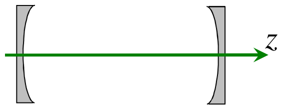


Unstable cavity

$$|\lambda_+| > 1 \text{ or } |\lambda_-| > 1$$

$$\mathbf{r}_n = \mathbf{M}_{RT}^n \mathbf{r}_0 = \mathbf{M}_{RT}^n (a\mathbf{r}_+ + b\mathbf{r}_-) = a\lambda_+^n \mathbf{r}_+ + b\lambda_-^n \mathbf{r}_-$$

$$\lambda_\pm = m \pm \sqrt{m^2 - 1} \text{ with } m = \frac{A+D}{2} \qquad |m| > 1$$

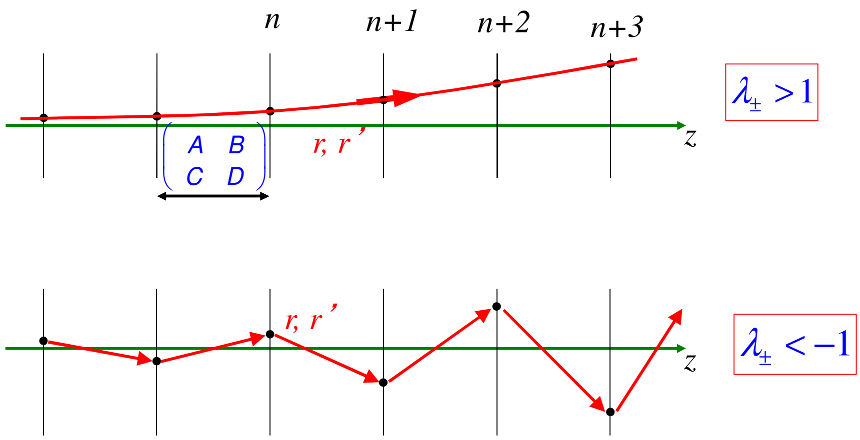
Let $m = \dfrac{A+D}{2} = \pm\cosh\theta$

$$\lambda_\pm = \cosh\theta \pm \sqrt{\cosh^2\theta - 1} = \cosh\theta \pm \sinh\theta = e^{\pm\theta}$$

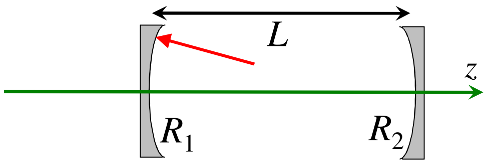
$$\mathbf{r}_n = a\lambda_+^n \mathbf{r}_+ + b\lambda_-^n \mathbf{r}_- = a\left(\pm e^{\theta}\right)^n \mathbf{r}_+ + b\left(\pm e^{-\theta}\right)^n \mathbf{r}_-$$

$$\mathbf{r}_n = \mathbf{r}_0 \cosh n\theta + \mathbf{s}_0 \sinh n\theta$$

Unstable cavity



Example: Cavity with two spherical mirrors



$$\mathbf{M}_{prop} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{M}_{mirror\_i} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R_i} & 1 \end{pmatrix}$$

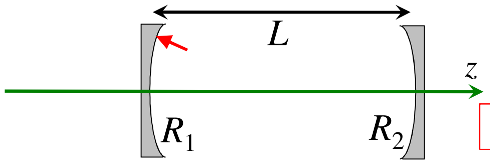
$$\mathbf{M}_{RT} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} =$$

$$\mathbf{M}_{RT} = \begin{pmatrix} \left(1-\frac{2L}{R_2}\right)\left(1-\frac{2L}{R_1}\right)-\frac{2L}{R_1} & L\left(1-\frac{2L}{R_2}\right)+L \\ -\frac{2}{R_2}\left(1-\frac{2L}{R_1}\right)-\frac{2}{R_2} & -\frac{2L}{R_2}+1 \end{pmatrix}$$

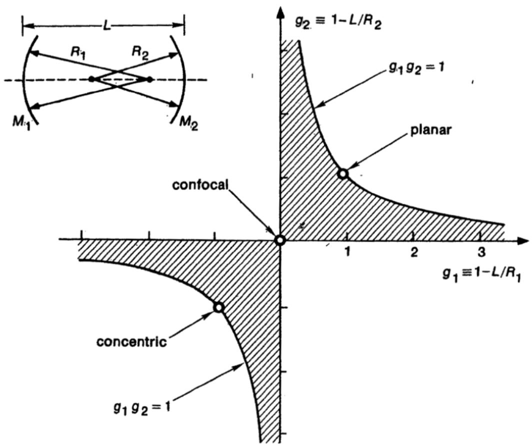
$$m = \frac{A+D}{2} = 2\left[1-\frac{L}{R_1}\right]\left[1-\frac{L}{R_2}\right]-1 \equiv 2g_1g_2-1$$

⇨ $-1 \le m \le 1$

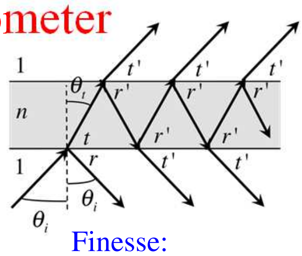
Example: Cavity with two spherical mirrors

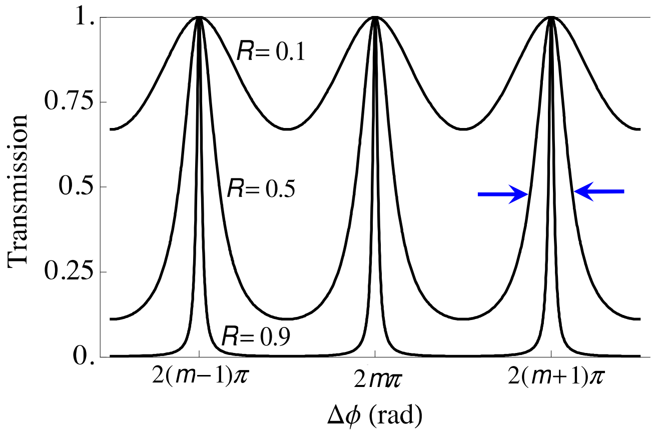


Stable cavity ⟷ $0 \le g_1 g_2 \le 1$



Fabry-Perot interferometer





$\frac{2\pi}{\mathcal{F}}$

Finesse:

$$\mathcal{F} = \frac{\pi\sqrt{R}}{1-R}$$

At resonance:

$$I_{\text{inside}} \simeq \frac{\mathcal{F}}{\pi} I_i$$

$$R = \left(\frac{n-1}{n+1}\right)^2$$
$$= 0.25 \text{ for } n = 3$$

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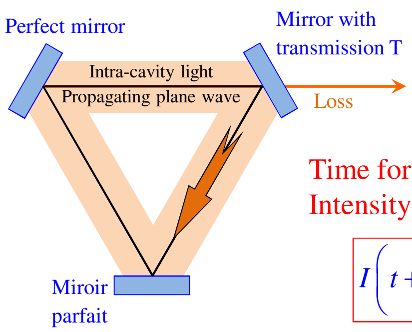
Photon lifetime in a high finesse cavity



- Cavity is filled with light (resonance)
- Turn off incident beam
- At $t = 0$, intensity $I(0)$ in the cavity

Time for 1 round trip in the cavity: L_{cav}/c
Intensity lost per round trip: $T\,I$

$$I\left(t + \frac{L_{cav}}{c}\right) = (1-T)\,I(t)$$

If the losses are low ($T << 1$):

$$I\left(t + \frac{L_{\text{cav}}}{c}\right) \simeq I(t) + \frac{L_{\text{cav}}}{c}\frac{dI}{dt}$$

$$\frac{dI}{dt} = -\frac{cT}{L_{cav}} I = -\frac{1}{\tau_{cav}} I$$

τ_{cav}: photon cavity lifetime.
It is as if the photons complete 1/T roundtrips in the cavity

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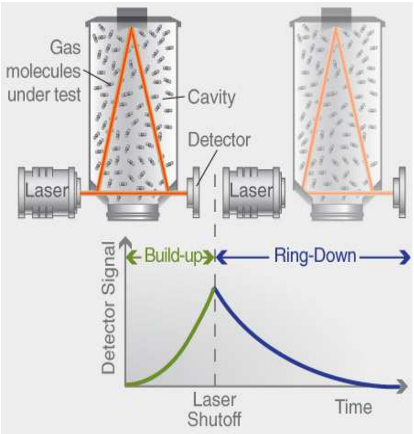
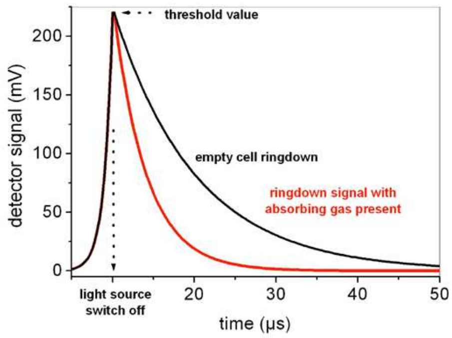
Cavity ring down spectroscopy



www.picarro.com



$$\tau_{\text{cav}} \simeq \frac{L_{\text{cav}}}{c \times \text{losses}} = \frac{L_{\text{cav}}}{c\left(1-R_1+1-R_2+1-R_3+\alpha L_{\text{cav}}\right)}$$

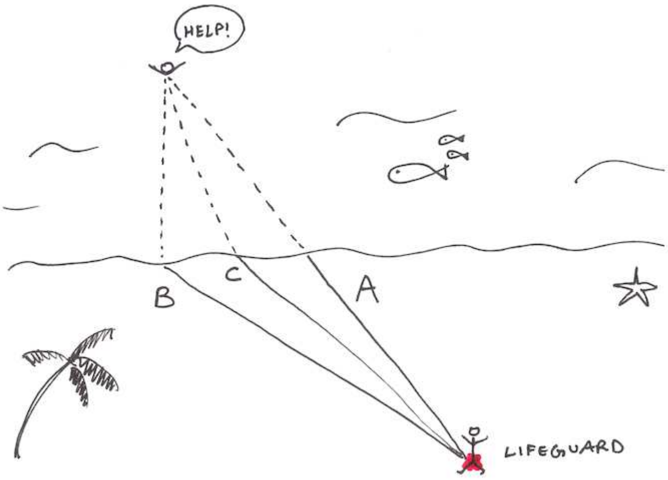
Eikonal equation: link between Maxwell's equation and ray optics

Consider the **general form of a propagating wave** (harmonic time dependence)

$$Y = A\exp\left[i\psi - i\omega t\right] + c.c.$$

Optical path length:

Shortest time to person drowning: optical path length



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Maxwell's equations

No free charges nor currents, simple media that may be ***inhomogeneous***

$$\varepsilon_r = \varepsilon_r(\vec{\mathbf{r}})$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (1.5)$$

$$\nabla \times \mathbf{B} = \varepsilon_0 \varepsilon_r \mu_0 \frac{\partial \mathbf{E}}{\partial t} \qquad (1.6)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (1.7)$$

$$\nabla \cdot (\varepsilon_0 \varepsilon_r \mathbf{E}) = 0 \qquad (1.8)$$

$$\varepsilon_r = n^2$$

n: index of refraction

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In search of the eikonal equation

Recall: $Y = A\exp[i\psi - i\omega t] + c.c.$

Define: $\mathcal{S}(\mathbf{r}) = \frac{\psi}{k_0}$

$$\Rightarrow \begin{cases} \mathbf{E}(\mathbf{r},t) = \boldsymbol{\mathcal{E}}(\mathbf{r})\exp[ik_0\mathcal{S}(\mathbf{r}) - i\omega t] \\ \mathbf{B}(\mathbf{r},t) = \boldsymbol{\mathcal{B}}(\mathbf{r})\exp[ik_0\mathcal{S}(\mathbf{r}) - i\omega t] \end{cases} \quad (1.1,\ 1.2)$$

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What is the eikonal?

$Y = A\exp[i\psi - i\omega t] + c.c.$

- The eikonal links geometrical and wave optics
- Eikonal : $\mathcal{S}(\mathbf{r}) = \frac{\psi}{k_0}$: spatial phase normalized by the wavevector in vacuum

$$\begin{cases} \mathbf{E}(\mathbf{r},t) = \boldsymbol{\mathcal{E}}(\mathbf{r})\exp[ik_0\mathcal{S}(\mathbf{r}) - i\omega t] \\ \mathbf{B}(\mathbf{r},t) = \boldsymbol{\mathcal{B}}(\mathbf{r})\exp[ik_0\mathcal{S}(\mathbf{r}) - i\omega t] \end{cases} \quad (1.1,\ 1.2)$$

- $\mathcal{S}(\mathbf{r}) =$ constant; surfaces of constant spatial phase, perpendicular to rays
- $\nabla\mathcal{S} = \|\nabla\mathcal{S}\|\ \hat{\mathbf{t}} = n(\mathbf{r})\hat{\mathbf{t}}$ in the direction of ray trajectories and energy flow
- $L_{opt} = \int_{M\to M'} n(x,y,z)ds = \int_{M\to M'} \nabla\mathcal{S}\bullet\frac{d\vec{\mathbf{r}}}{ds}ds = \mathcal{S}(M') - \mathcal{S}(M)$ Optical path length
- Approximation of geometrical optics: $\lambda_0 \to 0, i.e.,\ k_0 \to \infty$

  Amplitudes vary slowly as compared to the phase

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In search of the eikonal equation

Plan of attack: substitute (1.1), (1.2) into Maxwell's equations in order to find and expression for $\mathcal{S}(\mathbf{r})$

$$\begin{cases}\mathbf{E}(\mathbf{r},t)=\mathcal{E}(\mathbf{r})\exp[ik_0\mathcal{S}(\mathbf{r})-i\omega t]\\ \mathbf{B}(\mathbf{r},t)=\mathcal{B}(\mathbf{r})\exp[ik_0\mathcal{S}(\mathbf{r})-i\omega t]\end{cases} \quad (1.1,\ 1.2) \qquad \text{in} \qquad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \quad (1.5)$$

Recall: $\nabla\times(f\mathbf{A})=f\nabla\times\mathbf{A}+\nabla f\times\mathbf{A}$ $\qquad \frac{\omega}{k_0}=c$

$$\nabla\times\mathcal{E}+ik_0\nabla\mathcal{S}\times\mathcal{E}=i\omega\mathcal{B}$$

$$\frac{\nabla\times\mathcal{E}}{k_0}+i\nabla\mathcal{S}\times\mathcal{E}=\frac{i\omega\mathcal{B}}{k_0}=ic\mathcal{B}$$

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In search of the eikonal equation

Use the approximation of geometrical optics!

<u>Method 1</u>: consider $\lambda \ll$ all other dimensions, i.e., $\lambda_0\to 0,\ k_0\to\infty$

$$\frac{\nabla\times\mathcal{E}}{k_0}+i\nabla\mathcal{S}\times\mathcal{E}=ic\mathcal{B} \quad \Rightarrow \quad ik_0\nabla\mathcal{S}\times\mathcal{E}=i\omega\mathcal{B} \qquad (1.11)$$

<u>Method 2</u>: consider only situations where the amplitudes and ε_r vary slowly with distance as compared to the phase

$$\nabla\times\mathcal{E}+ik_0\nabla\mathcal{S}\times\mathcal{E}=i\omega\mathcal{B} \quad \Rightarrow \quad ik_0\nabla\mathcal{S}\times\mathcal{E}=i\omega\mathcal{B} \qquad (1.11)$$

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In search of the eikonal equation

Similary, substitute

$$\begin{cases}\mathbf{E}(\mathbf{r},t)=\mathcal{E}(\mathbf{r})\exp[ik_0\mathcal{S}(\mathbf{r})-i\omega t]\\ \mathbf{B}(\mathbf{r},t)=\mathcal{B}(\mathbf{r})\exp[ik_0\mathcal{S}(\mathbf{r})-i\omega t]\end{cases} \quad (1.1,\ 1.2)$$

into

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \quad (1.5)$$

$$\nabla\times\mathbf{B}=\varepsilon_0\varepsilon_r\mu_0\frac{\partial\mathbf{E}}{\partial t} \quad (1.6)$$

$$\nabla\cdot\mathbf{B}=0 \quad (1.7)$$

$$\nabla\cdot(\varepsilon_0\varepsilon_r\mathbf{E})=0 \quad (1.8)$$

And using the same approximation (geometrical optics) get:

$$ik_0\nabla\mathcal{S}\times\mathcal{E}=i\omega\mathcal{B} \quad (1.11)$$

$$ik_0\nabla\mathcal{S}\times\mathcal{B}=-i\omega\varepsilon_0\varepsilon_r\mu_0\mathcal{E} \quad (1.12)$$

$$\nabla\mathcal{S}\cdot\mathcal{B}=0 \quad (1.13)$$

$$\nabla\mathcal{S}\cdot\mathcal{E}=0 \quad (1.14)$$

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In search of the eikonal equation

$$ik_0\nabla\mathcal{S}\times\mathcal{E}=i\omega\mathcal{B} \quad (1.11)$$

$$ik_0\nabla\mathcal{S}\times\mathcal{B}=-i\omega\varepsilon_0\varepsilon_r\mu_0\mathcal{E} \quad (1.12)$$

$$\nabla\mathcal{S}\cdot\mathcal{B}=0 \quad (1.13)$$

$$\nabla\mathcal{S}\cdot\mathcal{E}=0 \quad (1.14)$$

Next, eliminate $\mathcal{B}$ from 1.12

Recall: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

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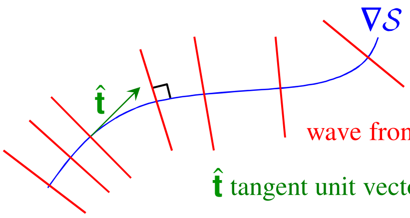
Eikonal equation

$$\left\|\nabla \mathcal{S}\right\|^2 = \varepsilon_r = n^2(\mathbf{r}) \qquad (1.15)$$

$$\nabla \mathcal{S} = \left\|\nabla \mathcal{S}\right\| \hat{\mathbf{t}} = n(\mathbf{r})\hat{\mathbf{t}} \qquad (1.16)$$

$\hat{\mathbf{t}}$: unit vector in the direction of $\nabla \mathcal{S}$



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The concept of a light ray

$$\frac{\omega}{k_0} = c$$

Recall: $\langle \mathbf{S} \rangle = \frac{2}{\mu_0} \mathbb{R}e\left[\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}}^*\right]$

$$ik_0 \nabla \mathcal{S} \times \boldsymbol{\mathcal{E}} = i\omega \boldsymbol{\mathcal{B}} \qquad (1.11)$$

Recall: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

$$\langle \mathbf{S} \rangle = \frac{2}{\mu_0}\left[\boldsymbol{\mathcal{E}} \times \frac{k_0}{\omega}\left(\nabla \mathcal{S} \times \boldsymbol{\mathcal{E}}^*\right)\right] = \frac{2}{c\mu_0}\left\|\boldsymbol{\mathcal{E}}\right\|^2 \nabla \mathcal{S} \qquad (1.19)$$

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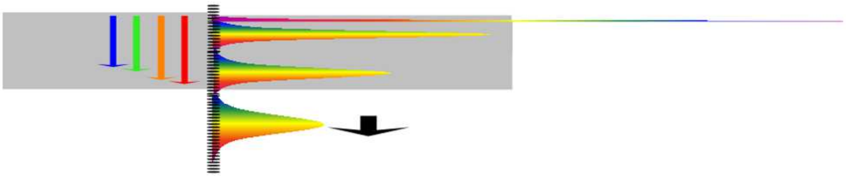
Lecture 3: Pulse propagation in dispersive media



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Goals for lecture 3

- Dispersion
  - Where does it come from?
  - What are its consequences?
- Propagation of pulses or wave packets in dispersive media

If we accept the electromagnetic theory of light, there is nothing left but to look for the cause of dispersion in the molecules of the medium itself.

Hendrik Lorentz (1878)
Nobel prize (1902)

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Introduction

- Dispersion occurs since

  a medium cannot respond instantaneously

  to an electromagnetic wave.

- The response of a material to an EM field must be
  causal i.e., it can depend on values of the field that existed in the past but not on those that will exist in the future!

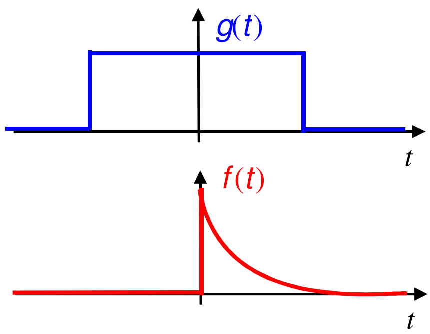
- Consequence: frequency dispersion and energy dissipation are intimately related.

Convolution:

$$[f(t) * g(t)](t) = \int_{-\infty}^{\infty} d\tau\, f(t-\tau)\, g(\tau)$$

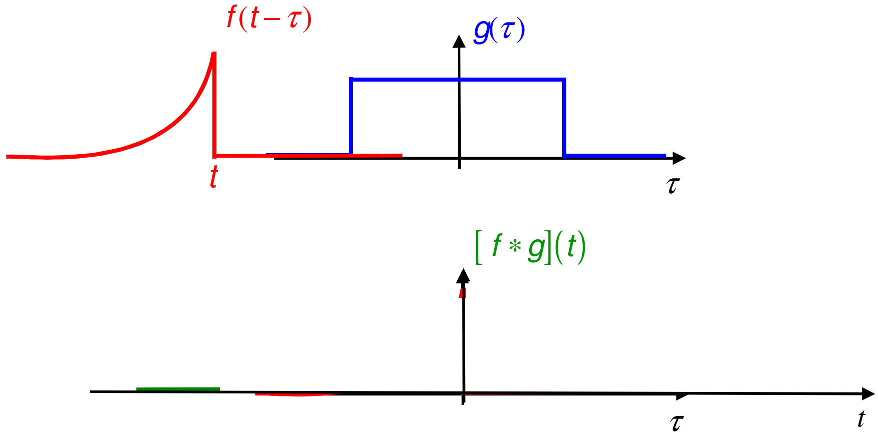
$g(t)$: excitation

$f(t)$: (non-instantaneous) impulse response function of the system



Convolution: excitation + system response (linear system)

$$\left[f(t)*g(t)\right](t)=\int_{-\infty}^{\infty} d\tau\, f(t-\tau)\,g(\tau)$$



What is the origin of dispersion?

Example: Conductivity $\quad \mathbf{j}=\sigma\mathbf{E}$

$$\mathbf{j}(\mathbf{r},t)=\int_{-\infty}^{t} dt'\,\sigma(t-t')\,\mathbf{E}(\mathbf{r},t') \qquad (4.1)$$

Causality: can only depend on values of the field that existed in the past

Let $\tau \equiv t-t'$

Build causality into the conductivity; define

$$\sigma(\tau)=0 \text{ for } \tau<0$$

$\sigma(\tau)\to 0 \text{ for } \tau\to-\infty$ Distant past has no influence.

$$\Rightarrow \quad \mathbf{j}(\mathbf{r},t)=\int_{-\infty}^{\infty} dt'\,\sigma(t-t')\,\mathbf{E}(\mathbf{r},t')$$

Fourier analysis

https://commons.wikimedia.org/wiki/File:Fourier_transform_time_and_frequency_domains.gif

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Fourier analysis for non-periodic functions

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)e^{ikx}\, dk$$

$$f(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\, dx$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, f(\omega)\, e^{-i\omega t}$$

$$f(\omega) = \int_{-\infty}^{\infty} dt\, f(t)\, e^{i\omega t}$$

Fourier transformations

Note: signs and position of 1/(2π) is a matter of convention.

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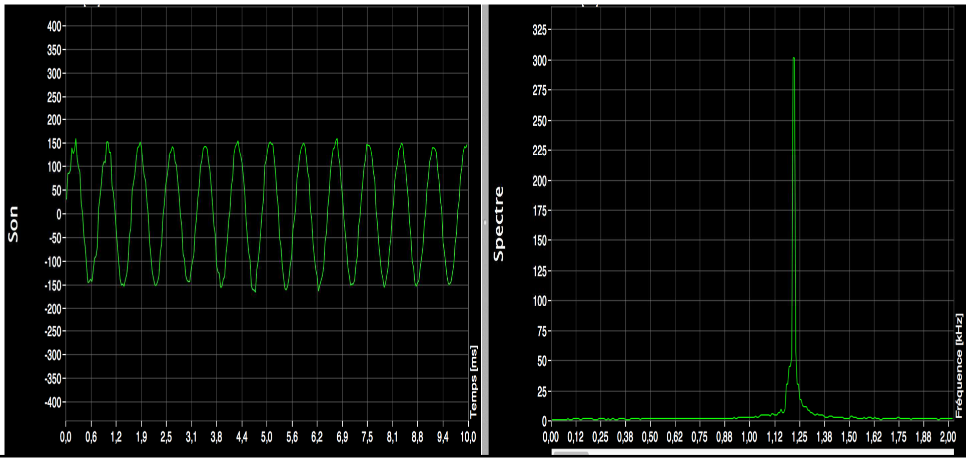
Fourier transforms

A physical interpretation

$|F(\omega)|^2$ is the power spectral density of the function $F(t)$, i.e., it tells us how much power is present at each frequency



Some properties of Fourier transforms

Derivatives and the Fourier transform

$$\frac{d}{dt}F(t) \rightleftharpoons -i\omega F(\omega)$$

$$itF(t) \rightleftharpoons \frac{d}{d\omega}F(\omega)$$

Fourier transform of a real function:

$$F(t) \in \mathbb{R} \Rightarrow F(-\omega) = \left[F(\omega)\right]^*$$

Some properties of Fourier transforms

$$F(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right) \longleftrightarrow F(\omega) = \sqrt{2\pi}\sigma \exp\left(-\frac{\omega^2\sigma^2}{2}\right)$$

$$\left(\frac{d}{dt}\right)^n F(t) \longleftrightarrow (-i\omega)^n F(\omega)$$

$$t^n F(t) \longleftrightarrow \left(-i\frac{d}{d\omega}\right)^n F(\omega)$$

$$F(at) \longleftrightarrow \frac{1}{|a|} F(\omega / a)$$

$$F(t+t_0) \longleftrightarrow e^{-i\omega t_0} F(\omega)$$

$$e^{i\omega_0 t} F(t) \longleftrightarrow F(\omega+\omega_0)$$

Some properties of Fourier transforms

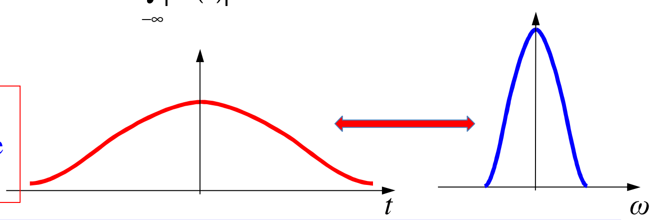
Widths

Suppose that for $F(t)$ and $F(\omega)$: $\int_{-\infty}^{\infty} F(t)\,dt = 0 \quad and \quad \int_{-\infty}^{\infty} F(\omega)\,d\omega = 0$

Suppose that $F(t)$ is normalized: $\int_{-\infty}^{\infty} |F(t)|^2\,dt = 1$

Then: $\Delta t \Delta\omega \geq \pi$

The larger $F(t)$ the thinner $F(\omega)$ and vice versa



PARSEVAL-PLANCHEREL theorem:

$$\int_{-\infty}^{\infty} F_1(t)\left[F_2(t)\right]^* dt = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} F_1(\omega)\left[F_2(\omega)\right]^* d\omega$$

Reciprocal spaces

Time ⟺ (**temporal**) frequency

Position ⟺ **spatial** frequency

Miguel Covarrubias, http://www.loc.gov/pictures/item/acd1996002431/PP/

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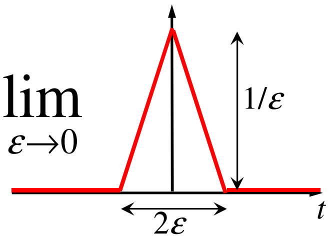
Dirac delta function

$$\delta(t) = \lim_{\varepsilon \to 0}$$



$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

Area = 1 $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

$$\delta(t) = \delta(-t)$$

$$\int_{-\infty}^{\infty} \delta(t) f(t)\,dt = f(0)$$

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = f(t_0)$$

Fourier transform:

$$1 = \int_{-\infty}^{\infty} \delta(t) e^{i\omega t}\,dt$$

$$\delta(t) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} e^{-i\omega t}\,d\omega$$

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Recall: Fourier transforms and the convolution integral

In general:

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(\omega)\, e^{-i\omega t} \quad \longleftrightarrow \quad f(\omega)=\int_{-\infty}^{\infty} dt\, f(t)\, e^{i\omega t}$$

$$\left[f(t)*g(t)\right](t)=\int_{-\infty}^{\infty} d\tau\, f(t-\tau)\, g(\tau) \quad \longleftrightarrow \quad f(\omega)\cdot g(\omega)$$

"The Fourier transform of a convolution integral is equal to the product of the Fourier transforms of the individual functions".

$$2\pi\, f(t)\cdot g(t) \quad \longleftrightarrow \quad \left[f(\omega)*g(\omega)\right](\omega)=\int_{-\infty}^{\infty} d\omega'\, f(\omega-\omega')\, g(\omega')$$

Fourier transform pairs

$$\hat{\sigma}(\omega)=\int_{-\infty}^{\infty}\sigma(t)e^{i\omega t}dt \qquad \sigma(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\sigma}(\omega)e^{-i\omega t}d\omega$$

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}d\omega$$

$$\mathbf{j}(\mathbf{r},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\mathbf{j}}(\mathbf{r},\omega)e^{-i\omega t}d\omega$$

Non-instantaneous response of a dispersive medium

$$\mathbf{j}(\mathbf{r},t) = \int_{-\infty}^{\infty} dt' \, \sigma(t-t') \mathbf{E}(\mathbf{r},t')$$

Fourier transform

$$\mathbf{j}(\mathbf{r},\omega) = \hat{\sigma}(\omega) \mathbf{E}(\mathbf{r},\omega)$$

$$\hat{\sigma}(\omega) = \sigma'(\omega) + i\sigma''(\omega)$$

What can you say about the symmetry of the real and imaginary parts of σ?

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Properties of the real and imaginary contributions

Recall: $\sigma(t)$ is real ⟹ $\sigma(t) = \sigma^*(t)$

From the definition of the Fourier transform:

$$\sigma(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\sigma'(\omega) + i\sigma''(\omega)] \, e^{-i\omega t} d\omega = \sigma^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\sigma'(\omega) - i\sigma''(\omega)] \, e^{+i\omega t} d\omega$$

Change ω to $-\omega$ in the second integral and conclude!

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Non-instantaneous response of a dispersive medium

$$\sigma(\tau)\in\mathbb{R}\Rightarrow\hat{\sigma}(-\omega)=\hat{\sigma}^{*}(\omega)\Rightarrow\begin{cases}\sigma'(\omega) & \text{even}\\ \sigma''(\omega) & \text{odd}\end{cases}$$

Same conclusions for $\hat{\sigma}(\omega),\hat{\mu}(\omega),\hat{\chi}(\omega),\hat{\varepsilon}(\omega)$
All are complex and depend on the frequency.

And what if dispersion didn't exist???

What happens if the response is the same for all frequences?

i.e., $\hat{\sigma}(\omega)\equiv\hat{\sigma}$

(inverse) Fourier transform

$$\sigma(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\sigma}e^{-i\omega t}d\omega$$

Recall: $\delta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{-i\omega t}$

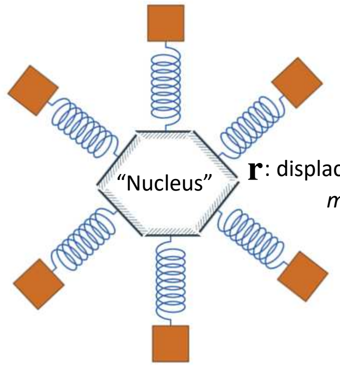
$$\sigma(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\sigma}e^{-i\omega t}d\omega=\frac{\hat{\sigma}}{2\pi}\underbrace{\int_{-\infty}^{\infty}e^{-i\omega t}\,d\omega}_{2\pi\delta(t)}=\hat{\sigma}\,\delta(t)$$

INSTANTANEOUS RESPONSE OF THE SYSTEM

Instantaneous response-- impossible! Frequency dispersion MUST exist

Next: Classical models for frequency dispersion: the Lorentz model

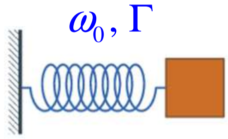
Basic idea: the movement of the (bound) electrons in a material is that of a *damped, driven, simple harmonic oscillator*



$\mathbf{r}$: displacement from equilibrium

m: electron mass

ω_0, Γ



ω_0 resonance frequency

Γ damping constant

$$m\frac{d^2\mathbf{r}}{dt^2} + m\Gamma\frac{d\mathbf{r}}{dt} + m\omega_0^2\mathbf{r} = -e\mathbf{E}$$

And for "free" electrons? ➡ No spring! ➡ Drude model for metals!!!

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Lorentz Model

$$m\frac{d^2\mathbf{r}}{dt^2} + m\Gamma\frac{d\mathbf{r}}{dt} + m\omega_0^2\mathbf{r} = -e\mathbf{E} \qquad (4.45)$$

Consider a steady state regime:

Monochromatic field $\mathbf{E}(t) = \mathcal{E}\exp(-i\omega t) + \text{c.c.}$ (4.46)

Expression for displacement: $\mathbf{r}(t) = \mathcal{R}\exp(-i\omega t) + \text{c.c.}$ (4.47)

Plug (4.46) and (4.47) in (4.45) and solve for $\mathcal{R}$!

$$\mathcal{R} = \frac{-e/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}\mathcal{E}$$

Dipole moment: $$\mathbf{p} = -e\mathbf{r}(t) = \frac{e^2/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}\mathcal{E}\exp(-i\omega t) + \text{c.c.}$$

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Lorentz Model

Dipole moment: $\mathbf{p} = -e\mathbf{r}(t) = \dfrac{e^2/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}\mathcal{E}\exp(-i\omega t) + \text{c.c.}$

Polarization (electric dipole moment per unit volume):

$$\mathbf{P} = n\mathbf{p}(t) = \frac{ne^2/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}\mathcal{E}\exp(-i\omega t) + \text{c.c.} \quad (4.49)$$

n: number of electrons per unit volume

Recall: $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon\mathbf{E}$

Use (4.49) in this expression

Let $\omega_p^2 = \dfrac{ne^2}{\varepsilon_0 m}$

Solve for $\hat{n}^2(\omega) = \dfrac{\hat{\varepsilon}(\omega)}{\varepsilon_0}$

This n is the index of refraction!

$$\hat{n}^2(\omega) = \frac{\hat{\varepsilon}(\omega)}{\varepsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}$$

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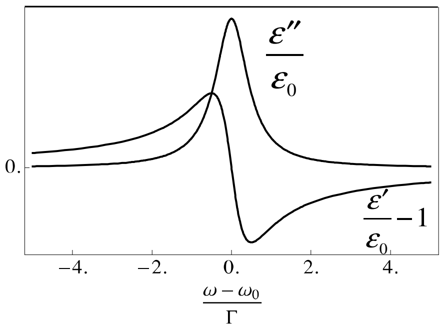
Lorentz model

Complex index of refraction

$$\hat{n}^2(\omega) = \frac{\hat{\varepsilon}(\omega)}{\varepsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}$$

$$\omega_p^2 = \frac{ne^2}{\varepsilon_0 m}$$

Real and imaginary parts



$$\frac{\varepsilon'(\omega)}{\varepsilon_0} = 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}$$

$$\frac{\varepsilon''(\omega)}{\varepsilon_0} = \frac{\omega_p^2\omega\Gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}$$

Almost Lorentzian if $\Gamma << \omega_0$

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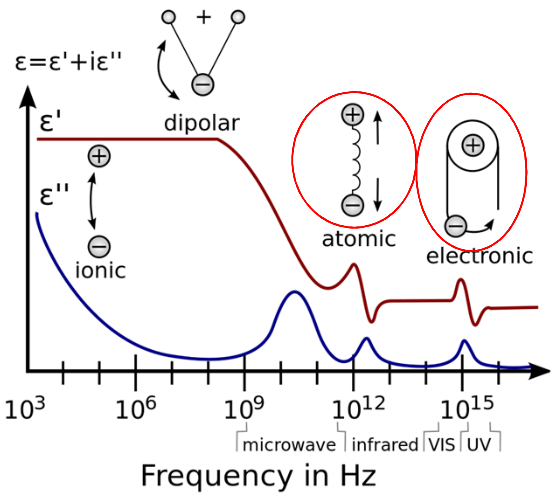
Dielectrics

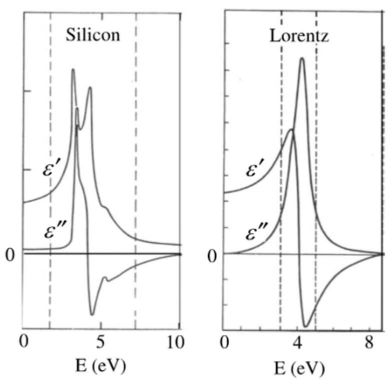
Frequency dependence of the permittivity

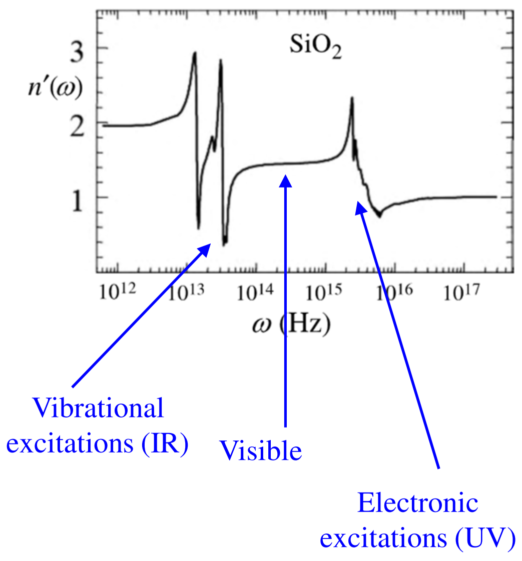


Wikipedia

Comparison of the Lorentz model with data

Silicon





Frequency dispersion and energy dissipation

$$\int_V d^3r\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)=-\int_V d^3r\,\mathbf{j}_l\cdot\mathbf{E}-\int_V d^3r\,\nabla\cdot(\mathbf{E}\times\mathbf{H})$$

Poynting theorem

+

$$\mathbf{D}(\mathbf{r},\omega)=\hat{\varepsilon}(\omega)\mathbf{E}(\mathbf{r},\omega)$$
$$\mathbf{B}(\mathbf{r},\omega)=\hat{\mu}(\omega)\mathbf{H}(\mathbf{r},\omega)$$

Lecture notes pp. 76-78 or Zangwill p. 627-629

⬇

$$\int_V d^3r\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)=\int_V d^3r\frac{\partial}{\partial t}u_{\mathrm{EM}}(t)+\int_V d^3r\,Q(t)$$

with

$$\begin{cases} u_{\mathrm{EM}}(t)=\dfrac{1}{2}\left\{\dfrac{\partial}{\partial\omega}\left[\omega\varepsilon'(\omega)\right]\left\|\mathbf{E}(t)\right\|^2+\dfrac{\partial}{\partial\omega}\left[\omega\mu'(\omega)\right]\left\|\mathbf{H}(t)\right\|^2\right\} \\ Q(t)=\omega\left[\varepsilon''(\omega)\left\|\mathbf{E}(t)\right\|^2+\mu''(\omega)\left\|\mathbf{H}(t)\right\|^2\right] \end{cases}$$

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Frequency dispersion and energy dissipation

$$\int_V d^3r\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)=\int_V d^3r\frac{\partial}{\partial t}u_{\mathrm{EM}}(t)+\int_V d^3r\,Q(t)$$

with

$$\begin{cases} u_{\mathrm{EM}}(t)=\dfrac{1}{2}\left\{\dfrac{\partial}{\partial\omega}\left[\omega\varepsilon'(\omega)\right]\left\|\mathbf{E}(t)\right\|^2+\dfrac{\partial}{\partial\omega}\left[\omega\mu'(\omega)\right]\left\|\mathbf{H}(t)\right\|^2\right\} \\ Q(t)=\omega\left[\varepsilon''(\omega)\left\|\mathbf{E}(t)\right\|^2+\mu''(\omega)\left\|\mathbf{H}(t)\right\|^2\right] \end{cases}$$

For quasi-monochromatic fields (i.e., strongly peaked around a single frequency—wave packet).

$u_{\mathrm{EM}}(t)$ is **the total energy per unit volume (transiently) stored in the medium**

$Q(t)$ is the **rate of energy absorption (per unit volume)**

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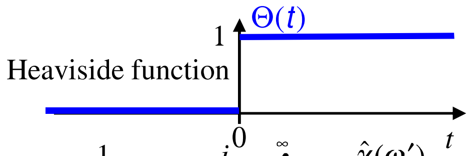
The consequences of causality: Kramers-Kronig relations

Recall: $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}; \quad \varepsilon = \varepsilon_0 (1+\chi)$

In order for causality to hold: $\mathbf{P}(\mathbf{r},t) = \varepsilon_0 \int_{-\infty}^{\infty} dt' \, \chi(t-t') \mathbf{E}(\mathbf{r},t')$

with $\boxed{\chi(\tau) = 0 \ \text{ for } \tau < 0}$

We can write $\boxed{\chi(t) = \Theta(t)\chi(t)}$



$$\hat{\chi}(\omega) = \frac{1}{2\pi} \hat{\Theta}(\omega) * \hat{\chi}(\omega)$$

$$\hat{\Theta}(\omega) = \pi\delta(\omega) + \text{p.v.}\frac{i}{\omega}$$

$$\Rightarrow \quad \hat{\chi}(\omega) = \frac{1}{2}\hat{\chi}(\omega) + \frac{i}{2\pi} \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\hat{\chi}(\omega')}{\omega - \omega'}$$

$$\chi'(\omega) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\chi''(\omega')}{\omega - \omega'}$$

$$\chi''(\omega) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\chi'(\omega')}{\omega - \omega'}$$

Kramers-Kronig relations

Link between real and imaginary parts

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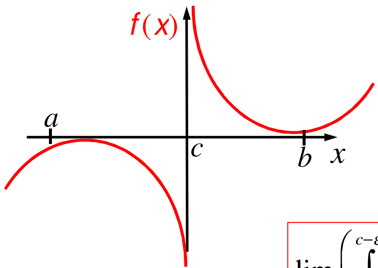
Cauchy principal value



$$\lim_{\varepsilon \to 0^+} \int_a^{c-\varepsilon} f(x)\,dx = -\infty$$

$$\lim_{\varepsilon \to 0^+} \int_{c+\varepsilon}^{b} f(x)\,dx = +\infty$$

$$\lim_{\varepsilon \to 0^+} \left(\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx \right) = \mathcal{P}\int_a^b f(x)\,dx$$

Cauchy principal value integral (if finite)

$$\text{p.v.}\frac{1}{x}: \quad \int_a^b \text{p.v.}\frac{1}{x} g(x)\,dx = \mathcal{P}\int_a^b \frac{g(x)}{x}\,dx$$

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Interpretation of the Kramers-Kronig relations

If $\varepsilon'(\omega)$ is not constant ***any***where, $\varepsilon''(\omega)$ is non-zero ***every***where

i.e., frequency dispersion in ***any*** interval of frequency implies that non-zero absorption occurs in ***every*** interval of frequency

Conversely, frequency dispersion occurs ***every***where in frequency if absorption occurs ***any***where in frequency.

Why is there this intimate relation between dispersion and energy dissipation?

$$\varepsilon = \varepsilon_0 (1+\chi)$$

$$\chi'(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\chi''(\omega')}{\omega-\omega'}$$

$$\chi''(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\chi'(\omega')}{\omega-\omega'}$$

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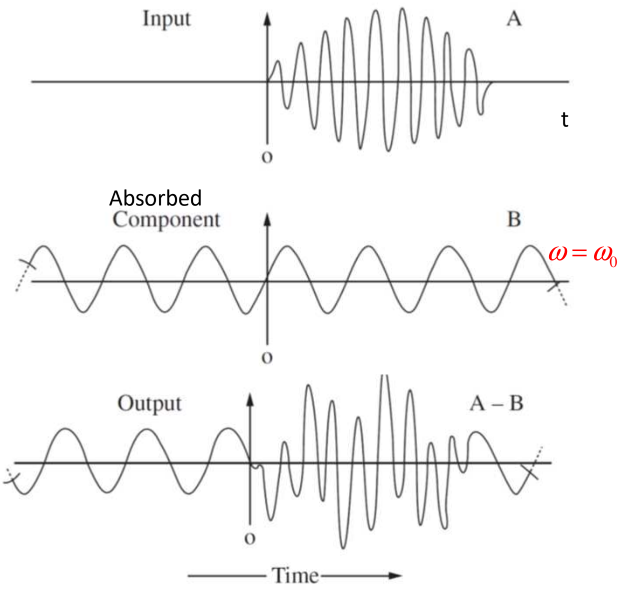
Frequency dispersion, absorption and causality

Wave packet in causal medium

Assume material absorbs only for $\omega=\omega_0$

Result: signal before t=0 !!!



Zangwill, p.651

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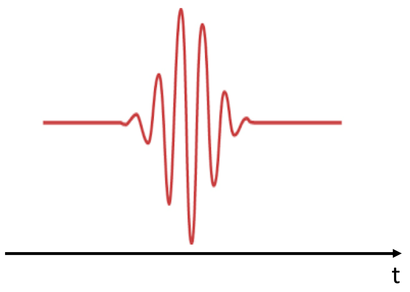
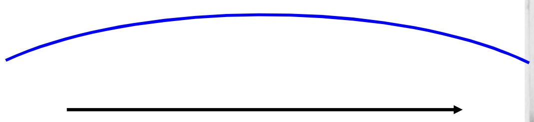
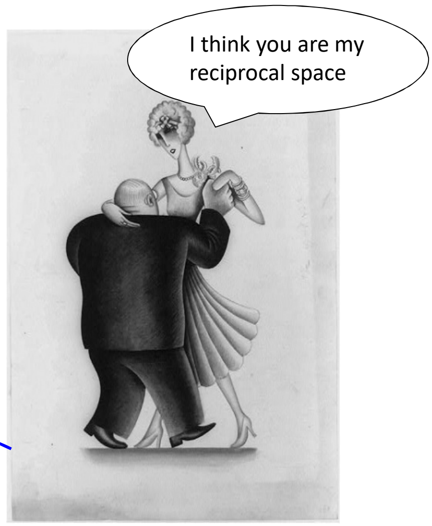
Short light pulses and consequences of dispersion

Pulse as a function of time



Frequency space





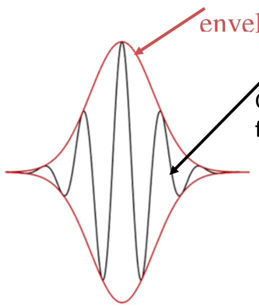
Pulse propagation

$$E(z=0,t)=\mathcal{E}(z=0,t)\,e^{-i\omega_p t}+\text{c.c.}=E^{(+)}(z=0,t)+\text{c.c.}$$

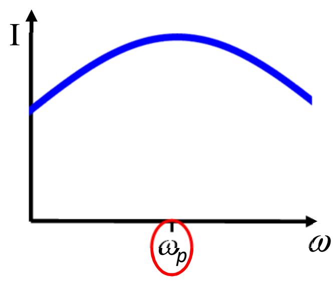
Analytic signal

envelope

Central frequency



Pulse as a function of time



Frequency space

Pulse propagation

$$E(z=0,t)=\mathcal{E}(z=0,t)e^{-i\omega_p t}+\text{c.c.}=E^{(+)}(z=0,t)+\text{c.c.}$$

Example: Gaussian envelope $\mathcal{E}(z=0,t)=E_0\exp\left(-\frac{t^2}{2\Delta t_0^2}\right)$

$$E^{(+)}(z=0,t)=E_0\exp\left(-\frac{t^2}{2\Delta t_0^2}\right)e^{-i\omega_p t}$$

↓ Fourier transform

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Pulse propagation

$$E(z=0,t)=\underbrace{\mathcal{E}(z=0,t)}e^{-i\omega_p t}+\text{c.c.}=E^{(+)}(z=0,t)+\text{c.c.}$$

Example: Gaussian envelope $\mathcal{E}(z=0,t)=E_0\exp\left(-\frac{t^2}{2\Delta t_0^2}\right)$

$$E^{(+)}(z=0,t)=E_0\exp\left(-\frac{t^2}{2\Delta t_0^2}\right)e^{-i\omega_p t}$$

FT

$$E^{(+)}(z=0,\omega)=\hat{E}_0\exp\left(-\frac{(\omega-\omega_p)^2}{2\Delta\omega^2}\right)$$

$\hat{E}_0=\sqrt{2\pi}\Delta t_0 E_0$

$$\boxed{\Delta\omega=1/\Delta t_0}$$

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Pulse propagation

As we will see when we study diffraction, we can write an expression for the pulse at the entrance to the dispersive medium ***as a sum of monochromatic plane waves***

$$E^{(+)}(z=0,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}E^{(+)}(z=0,\omega)$$

Each component propagates with its own wave number $k(\omega)=n(\omega)\frac{\omega}{c}$

$$E^{(+)}(z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}E^{(+)}(z=0,\omega)\,e^{ik(\omega)z} \quad (4.73)$$

Assume wave packet is sharply peaked at ω_p

→ Taylor's expansion of $k(\omega)$ around ω_p (to second order)

Dispersion: $$k(\omega)=n(\omega)\frac{\omega}{c}\simeq k(\omega_p)+(\omega-\omega_p)\left.\frac{dk}{d\omega}\right|_{\omega_p}+\frac{1}{2}(\omega-\omega_p)^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_p}$$

Pulse propagation

Dispersion: $$k(\omega)=n(\omega)\frac{\omega}{c}\simeq k(\omega_p)+(\omega-\omega_p)\left.\frac{dk}{d\omega}\right|_{\omega_p}+\frac{1}{2}(\omega-\omega_p)^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_p} \quad (4.75)$$

$\left.\frac{dk}{d\omega}\right|_{\omega_p}$ → $1/v_g$; $\left.\frac{d^2k}{d\omega^2}\right|_{\omega_p}$ → β_2

$$k(\omega)=n(\omega)\frac{\omega}{c}$$

$$v_g=\frac{c}{n+\omega\frac{dn}{d\omega}}=\frac{c}{n-\lambda\frac{dn}{d\lambda}} \quad (4.76)$$

Group velocity

$$\beta_2=\left.\frac{d^2k}{d\omega^2}\right|_{\omega_p} \quad (4.77)$$

Group velocity dispersion

Substitute (4.75-4.77) in (4.73), evaluate integral!!!

$$E^{(+)}(z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}E^{(+)}(z=0,\omega)\,e^{ik(\omega)z} \quad (4.73)$$

Propagation of a Gaussian pulse

$$E^{(+)}(z=0,\omega)=\hat{E}_0\exp\left(-\frac{(\omega-\omega_p)^2}{2\Delta\omega^2}\right)$$

$$E^{(+)}(z=0,t)=E_0\exp\left(-\frac{t^2}{2\Delta t_0^2}\right)e^{-i\omega_p t}$$

$$\beta_2=\left.\frac{d^2k}{d\omega^2}\right|_{\omega_p}$$

Propagation in a dispersive medium

Phase velocity

$$v_\varphi=\frac{\omega_p}{k_p}\qquad k_p\equiv k(\omega_p)$$

Envelope propagates at the group velocity v_g

$$E^{(+)}(z,t)\propto E_0 e^{-i(\omega_p t-k_p z)}\times\exp\left[-\frac{(t-z/v_g)^2}{2\Delta t(z)^2}\right]\exp\left[-i\frac{(t-z/v_g)^2}{2\Delta t(z)^2}\beta_2\Delta\omega^2 z\right]$$

$\beta_2\neq 0$: pulse spreading

$\beta_2\neq 0$: chirp

$$\Delta t(z)^2=\Delta t_0^2+\Delta\omega^2\beta_2^2 z^2$$

$$E^{(+)}(z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega e^{-i\omega t}E^{(+)}(z=0,\omega)\,e^{ik(\omega)z}$$

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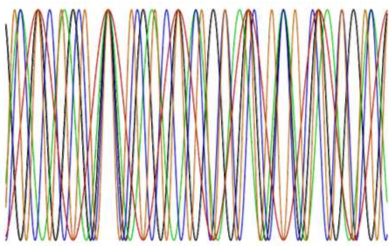
Phase and group velocities

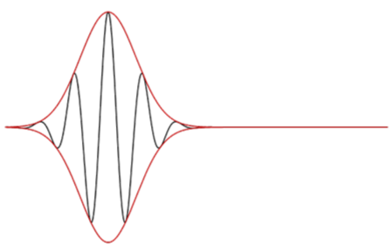
✓ Phase velocity:

$$v_\varphi=\frac{\omega_p}{k_p}=\frac{c}{n(\omega_p)}$$



✓ Group velocity:

$$v_g=\frac{c}{n+\omega\dfrac{dn}{d\omega}}$$



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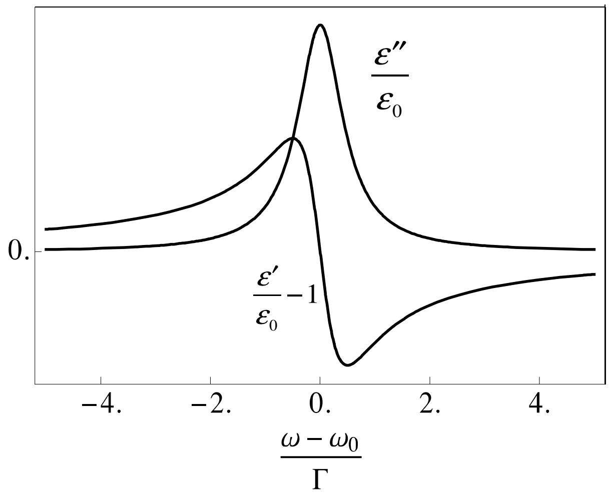
Normal dispersion

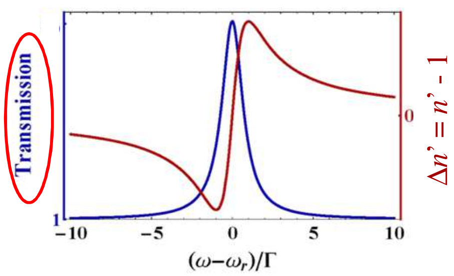


$$\frac{dn}{d\omega} > 0$$

$$v_g = \frac{c}{n + \omega \frac{dn}{d\omega}}$$

Normal dispersion and slow light

Kramers-Kronig relations: link between dispersion and absorption



Peak in *transmission*:

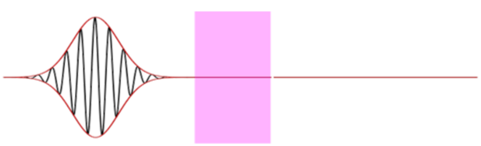
↓

Normal dispersion--
positive slope

↓

Slow light

$$v_g = \frac{c}{n + \omega \frac{dn}{d\omega}} \ll c$$



Normal dispersion and slow light

Light speed reduction to 17 metres per second in an ultracold atomic gas

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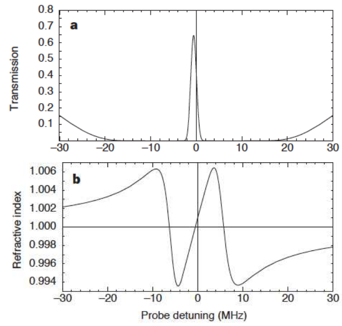
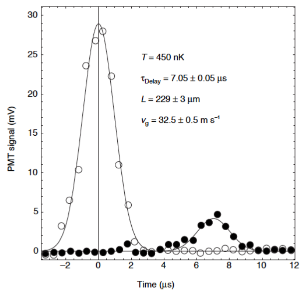
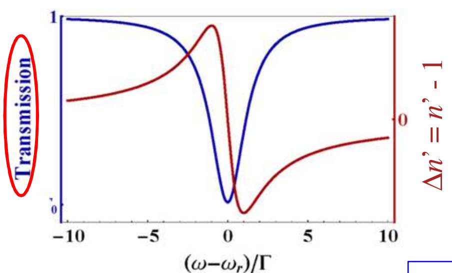




Anomolous dispersion and fast light

Kramers-Kronig relations: link between dispersion and absorption



Peak in *absorption*:

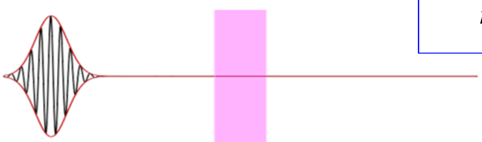
↓

Anomolous dispersion--negative slope

↓

Fast light

$$v_g = \frac{c}{n + \omega \frac{dn}{d\omega}} > c \quad \text{or even } < 0$$



Anomolous dispersion and fast light ($v_g > c$)

The speed of information in a 'fast-light' optical medium

NATURE | VOL 425 | 16 OCTOBER 2003 | www.nature.com/nature

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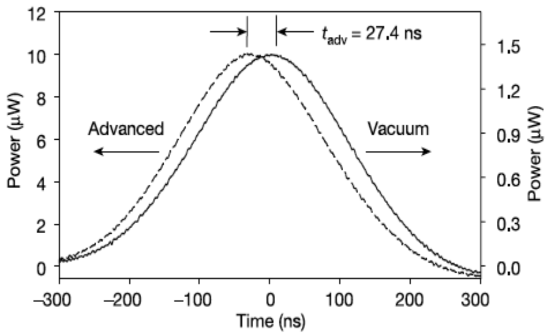
[2]*Department of Electrical and Computer Engineering, The Optical Sciences Center, University of Arizona, Tucson, Arizona 85721, USA*

- What about causality?
- Can you transmit *information* faster than *c*?



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Anomolous dispersion and fast light ($v_g > c$)

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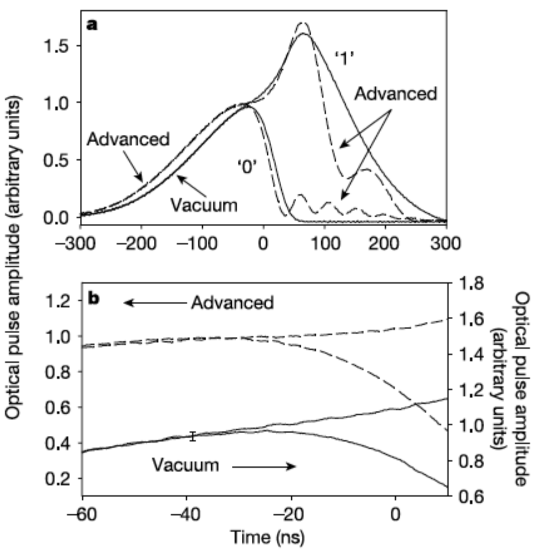
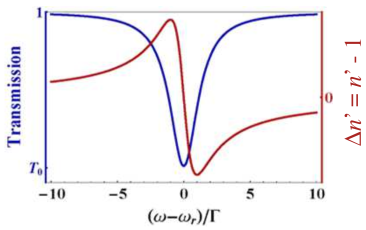
What about causality?



At what moment can you distinguish a 1 from a 0?



A step edge contains a multitude of frequencies!

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Propagation of a Gaussian pulse

Phase velocity

$v_\varphi = \dfrac{\omega_p}{k_p}$

Envelop propagates at the group velocity v_g

$$E^{(+)}(z,t) \propto E_0 e^{-i(\omega_p t - k_p z)} \times \exp\left[-\frac{(t - z/v_g)^2}{2\Delta t(z)^2}\right] \exp\left[-i\frac{(t - z/v_g)^2}{2\Delta t(z)^2}\beta_2 \Delta\omega^2 z\right]$$

$\beta_2 \neq 0$: pulse spreading

$\beta_2 \neq 0$: chirp

$$\Delta t(z)^2 = \Delta t_0^2 + \Delta\omega^2 \beta_2^2 z^2$$

Consequences of $\beta_2 \neq 0$:

- Pulse spreads out in time
- Pulse becomes "chirped": instantaneous frequency varies linearly with time

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Chirp: instantaneous frequency varies linearly with time

$$E^{(+)}(z,t) \propto E_0 e^{-i(\omega_p t - k_p z)} \times \exp\left[-\frac{(t - z/v_g)^2}{2\Delta t(z)^2}\right] \exp\left[-i\frac{(t - z/v_g)^2}{2\Delta t(z)^2}\beta_2 \Delta\omega^2 z\right]$$

$$E^{(+)}(t) \propto e^{-at^2} e^{-i(\omega_p t + bt^2)} \qquad \phi_{tot} = \omega_p t + bt^2$$

Instantaneous frequency: $\omega_i \equiv \dfrac{d\phi_{tot}}{dt} = \omega_p + 2bt$

β_2

Recall: $$k(\omega) = n(\omega)\frac{\omega}{c} \simeq k(\omega_p) + (\omega - \omega_p)\left.\frac{dk}{d\omega}\right|_{\omega_p} + \frac{1}{2}(\omega - \omega_p)^2 \left.\frac{d^2k}{d\omega^2}\right|_{\omega_p}$$

$\beta_2 > 0$: high frequencies propagate more slowly than the low frequencies

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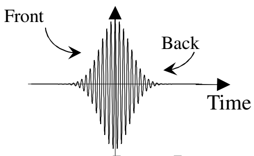
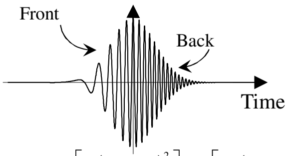
Pulse spreading as a function of the initial pulse width



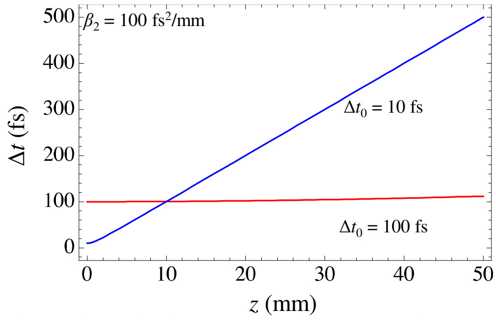
$n(\omega)$



$$E^{(+)}(z=0,t)=E_0\exp\left[-\frac{t^2}{\Delta t_0^2}\right]e^{-i\omega_p t}$$

$$E^{(+)}(z,t)\propto E_0 e^{-i(\omega_p t-k_p z)}\times\exp\left[-\frac{(t-z/v_g)^2}{2\Delta t(z)^2}\right]\exp\left[-i\frac{(t-z/v_g)^2}{2\Delta t(z)^2}\beta_2\Delta\omega^2 z\right]$$

$$\Delta t(z)^2=\Delta t_0^2+\Delta\omega^2\beta_2^2 z^2$$



The shorter the pulse, the larger the spectrum, the more it will spread!!!

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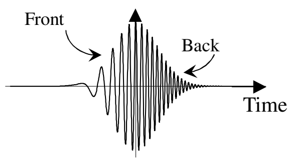
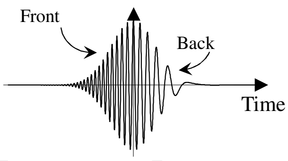
Pulse spreading, pulse compression…

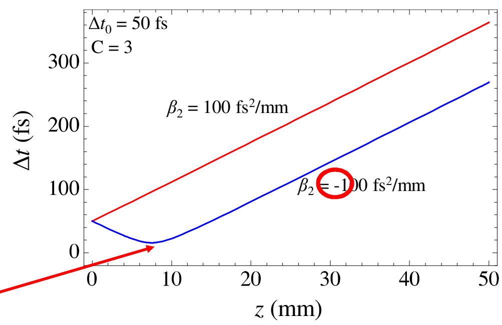


$n(\omega)$



$$E^{(+)}(z=0,t)=E_0\exp\left[-\frac{1+iC}{2}\frac{t^2}{\Delta t_0^2}\right]e^{-i\omega_p t}$$

Initial chirp



Pulse width decrease!

More in tutorial!

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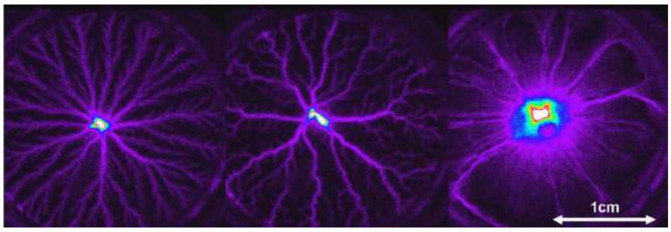
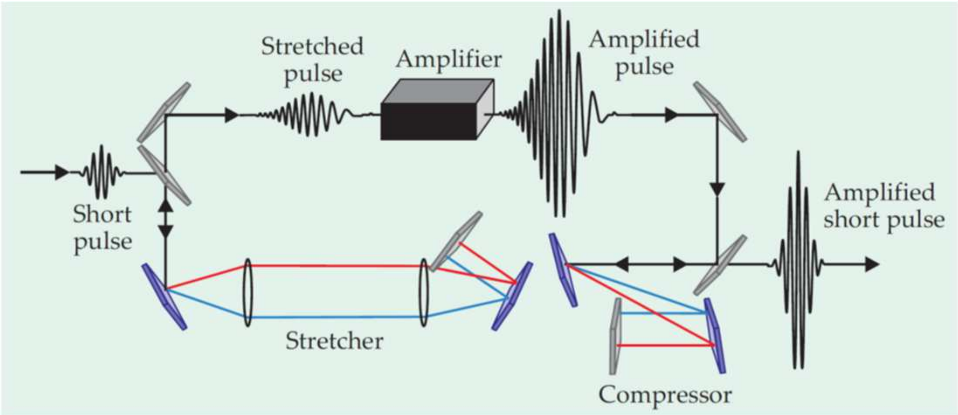
Chirped pulse amplification:
2018 Nobel Prize in Physics

Why do you need Chirped Pulse Amplification?



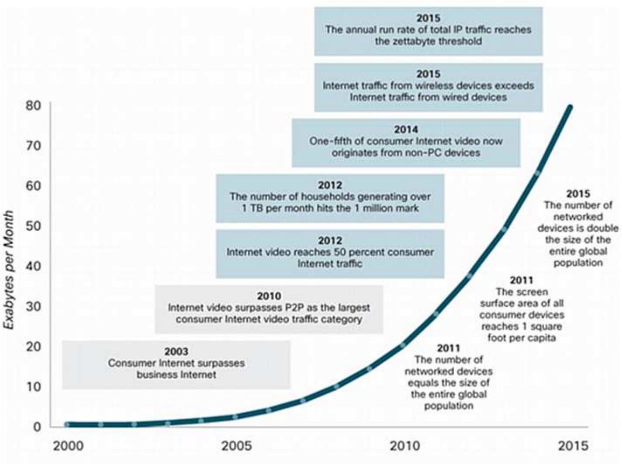
Too high power in gain medium! Plasma filamentation!

Chirped pulse amplification:
2018 Nobel Prize in Physics



The information highway = optical fibres = dispersive medium!

Figure 1. Five Traffic Milestones and Three Traffic Generator Milestones by 2015



Source: Cisco VNI, 2011

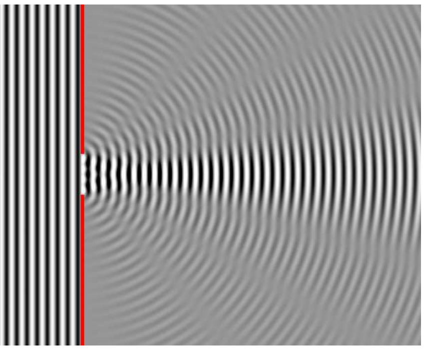
Summary

- Dispersion: optical response depends on frequency of light
- Origin of dispersion: the material cannot respond instantaneously!
- Consequences of dispersion:
  - light pulse « shaping » (most often spreading…)
  - Energy dissipation in medium
- Speed(s) of light in matter: can have $0 < v_g < c$, $v_g > c$, $v_g < 0$.

Speed of *signal* is always in agreement with special relativity.

Lecture 4: Diffraction

Knowing the distribution of the electric field on a plane at z=0 (e.g. on an aperture), can we find an expression for the field at a distance z>0?

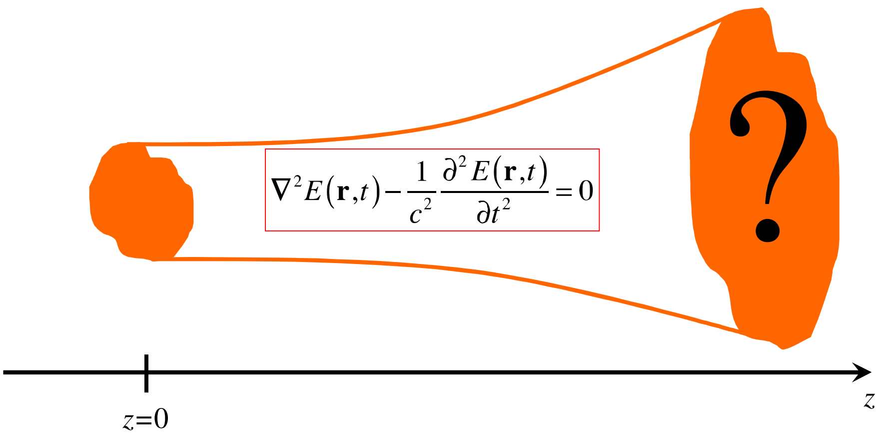


https://en.wikipedia.org/wiki/Huygens%E2%80%93Fresnel_principle

Why study diffraction?

https://en.wikipedia.org/wiki/Diffraction

Knowing the distribution of the electric field on a plane at z=0 can we find an expression for the field at a distance z>0?

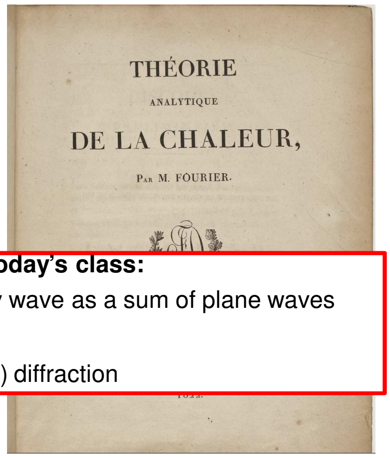


$$\nabla^2 E(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 E(\mathbf{r},t)}{\partial t^2} = 0$$

In principle, quite complicated: partial differential equation, boundary conditions, vector fields…

Solution!

Joseph Fourier (1768-1830)



Goals for the rest of today's class:

- Express an arbitrary wave as a sum of plane waves
- Spatial frequencies
- Fraunhofer (far-field) diffraction

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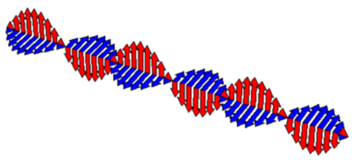
The electric field and its Fourier transform



The electric field is a function of $E(\mathbf{r},t)$

$$E(\mathbf{r},\omega) = \int_{-\infty}^{\infty} dt\, E(\mathbf{r},t)\, e^{i\omega t} \quad (3.2)$$

$$E(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, E(\mathbf{r},\omega)\, e^{-i\omega t} \quad (3.3)$$

Fourier transform with respect to time / angular frequency

$$E(k_x,k_y,z,t) = \iint dxdy E(x,y,z,t)\, e^{-i(k_x x + k_y y)}$$

$$E(x,y,z,t) = \frac{1}{(2\pi)^2}\iint dk_x dk_y E(k_x,k_y,z,t)\, e^{i(k_x x + k_y y)}$$

Fourier transform with respect to position / spatial frequency

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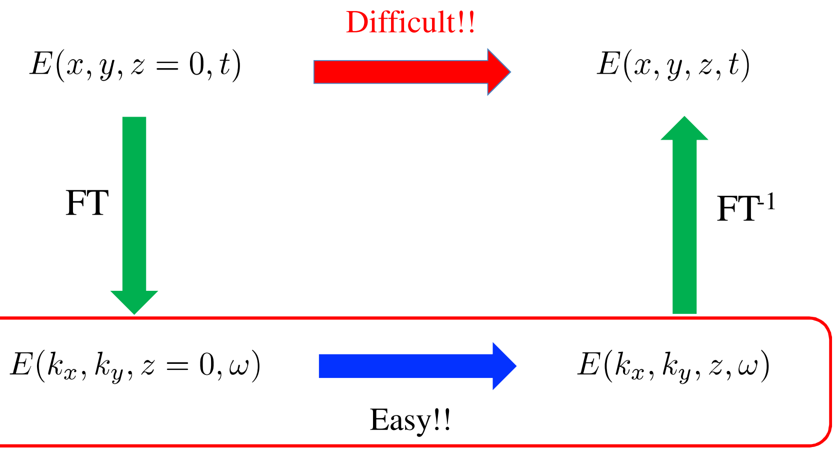
Diffraction and propagation of a light beam:
How will we get to our goal?



Wave propagation equation

- Consider one component of the electric field, i.e., a scalar function (good approx. when not near aperture).

Recall:

Maxwell's equations in vacuum (no free charges, no free currents)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1)$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (2)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (3)$$

$$\nabla \cdot \mathbf{E} = 0 \quad (4)$$

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

$$\nabla^2 E(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 E(\mathbf{r}, t)}{\partial t^2} = 0 \quad (3.1)$$

Helmholtz equation

$$\nabla^2 E(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 E(\mathbf{r},t)}{\partial t^2} = 0 \qquad (3.1)$$

Express $E(\mathbf{r},t)$ as a Fourier series or transform with respect to time / angular frequency

$$E(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega E(\mathbf{r},\omega)e^{-i\omega t} \qquad (3.3)$$

Plug (3.3) into (3.1)

$$\nabla^2 E(\mathbf{r},\omega) + \frac{\omega^2}{c^2} E(\mathbf{r},\omega) = 0 \qquad (3.4)$$

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Towards a propagating wave as a sum of plane waves

$$E(x,y,z,\omega) = \frac{1}{(2\pi)^2}\iint dk_x dk_y E(k_x,k_y,z,\omega)e^{i(k_x x + k_y y)} \qquad (3.5)$$

$$\nabla^2 E(\mathbf{r},\omega) + \frac{\omega^2}{c^2} E(\mathbf{r},\omega) = 0 \quad (3.4)$$

Plug (3.5) into (3.4)! Recall: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

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Towards a propagating wave as a sum of plane waves

$$\left[\frac{\partial^2}{\partial z^2}E(k_x,k_y,z,\omega)+\left(\frac{\omega^2}{c^2}-k_x^2-k_y^2\right)E(k_x,k_y,z,\omega)\right]=0 \qquad (3.6)$$

$$k_z=\begin{cases}\sqrt{\dfrac{\omega^2}{c^2}-k_x^2-k_y^2} & \text{if} \quad \dfrac{\omega^2}{c^2}>k_x^2+k_y^2 \\ i\sqrt{k_x^2+k_y^2-\dfrac{\omega^2}{c^2}} & \text{otherwise}\end{cases} \qquad (3.8)$$

Towards a propagating wave as a sum of plane waves

$$E\left(k_x,k_y,0,\omega\right)=A\left(k_x,k_y,\omega\right) \qquad \text{(3.7b) for } z=0$$

Recall: $E\left(k_x,k_y,z,\omega\right)=A\left(k_x,k_y,\omega\right)e^{ik_z z}$ (3.7b)

Recall: $E\left(x,y,z,\omega\right)=\left(\frac{1}{2\pi}\right)^2\iint dk_x dk_y E(k_x,k_y,z,\omega)e^{i(k_x x+k_y y)}$ (3.5)

$$E\left(x,y,z,\omega\right)=\frac{1}{\left(2\pi\right)^2}\iint dk_x dk_y E\left(k_x,k_y,z=0,\omega\right)e^{i\left(k_x x+k_y y+k_z z\right)} \quad (3.11)$$

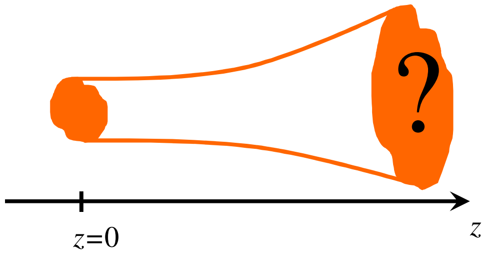
$$k_x^2+k_y^2+k_z^2=\frac{\omega^2}{c^2} \quad \text{(dispersion relation)} \quad (3.12)$$

Propagation and spatial frequencies

$$E(x,y,z,\omega)=\frac{1}{(2\pi)^2}\iint dk_x dk_y E(k_x,k_y,z=0,\omega)e^{i(k_x x+k_y y+k_z z)} \quad (3.11)$$

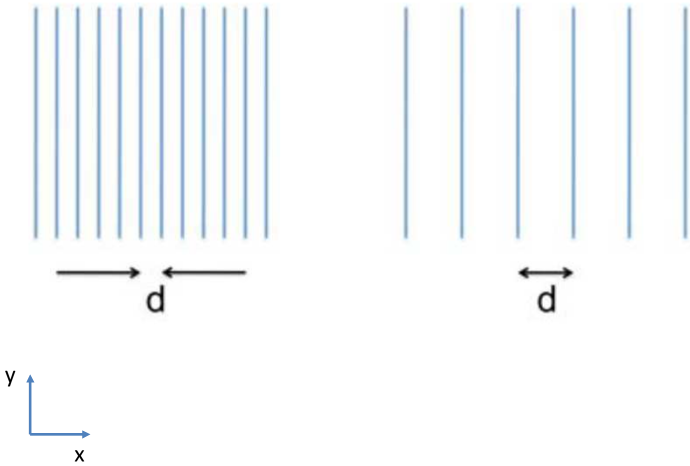
$$k_x^2+k_y^2+k_z^2=\frac{\omega^2}{c^2}$$

$$E(k_x,k_y,z=0,\omega)=\iint dxdy E(x,y,z=0,\omega)e^{-i(k_x x+k_y y)} \quad (3.13)$$



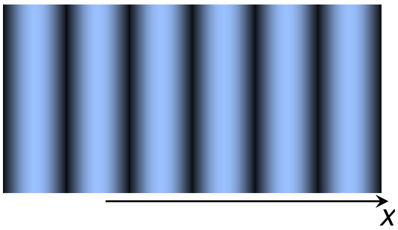
Concept of spatial frequencies

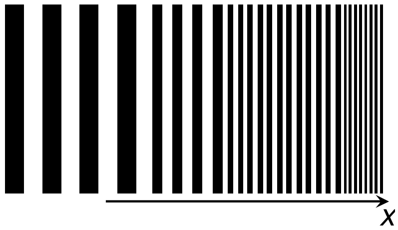
Which grating has the higher spatial frequency components?

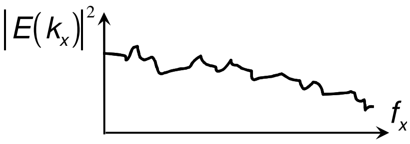


Concept of spatial frequencies

Which image has the most spatial frequency components?





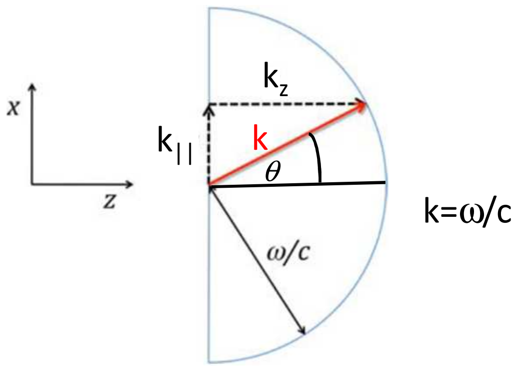


Spatial frequencies and propagation direction

- After propagation, the resulting field may be expressed as a sum (integral) of plane waves with wavevectors $k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$

$$E(x,y,z,\omega) = \frac{1}{(2\pi)^2} \iint dk_x dk_y E(k_x,k_y,z=0,\omega) e^{i(k_x x + k_y y + k_z z)}$$

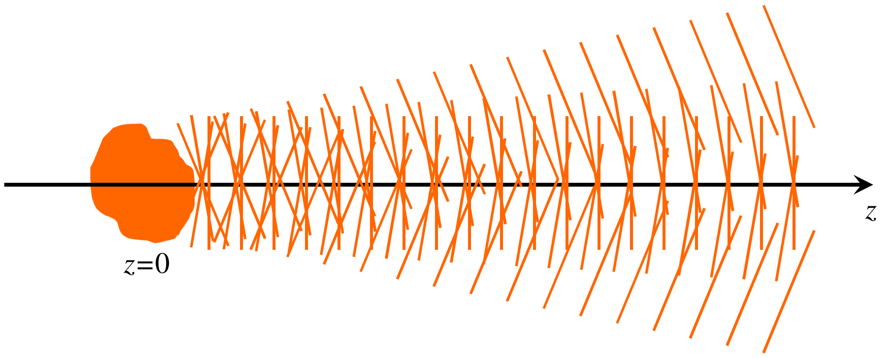
- *Each plane wave in the sum corresponds to a specific* ***spatial frequency***
- *Each plane wave in the sum corresponds to a specific propagation direction!*



Spatial frequencies and propagation direction

$$E(x,y,z,\omega)=\frac{1}{(2\pi)^2}\iint dk_x dk_y E(k_x,k_y,z=0,\omega)\, e^{i(k_x x+k_y y+k_z z)}$$

$$E(k_x,k_y,z=0,\omega)=\iint dxdy E(x,y,z=0,\omega)\, e^{-i(k_x x+k_y y)}$$



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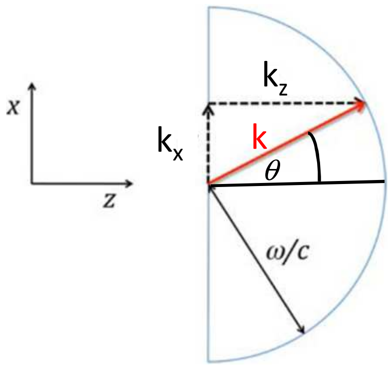
Divergence

From the properties of Fourier transform pairs:

$$\Delta x \Delta k_x > 1$$



$$\Delta\theta \approx \frac{\lambda}{\Delta x}$$

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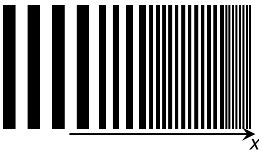
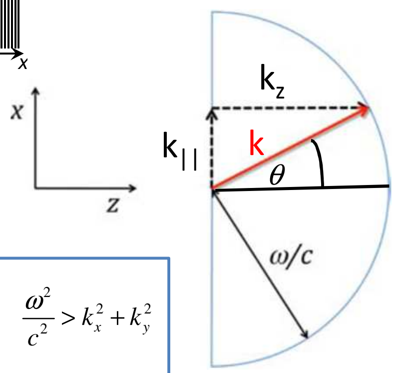
Propagation as a low-pass filter



$$E(x,y,z,\omega)=\frac{1}{(2\pi)^2}\iint dk_x dk_y E(k_x,k_y,z=0,\omega)e^{i(k_x x+k_y y+k_z z)}$$

$$k_x^2+k_y^2+k_z^2=\frac{\omega^2}{c^2}$$



$$k_z=\begin{cases}\sqrt{\frac{\omega^2}{c^2}-k_x^2-k_y^2} & \text{if } \frac{\omega^2}{c^2}>k_x^2+k_y^2\\ i\sqrt{k_x^2+k_y^2-\frac{\omega^2}{c^2}} & \text{otherwise}\end{cases}$$

k=ω/c

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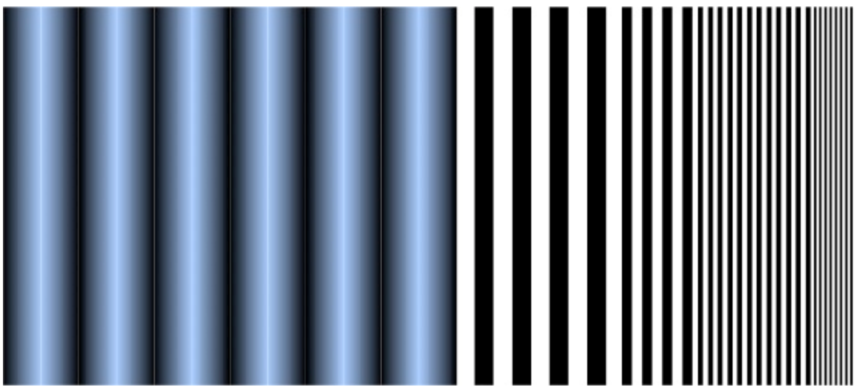
Propagation and spatial frequencies

$z=0$



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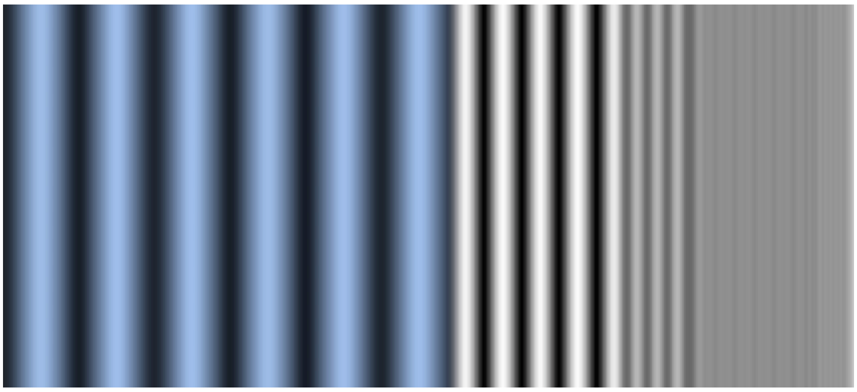
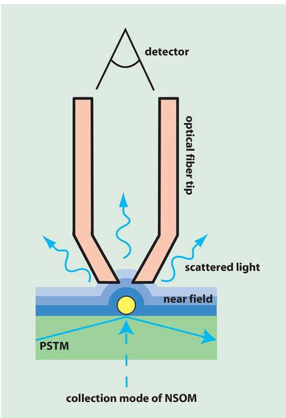
Propagation and spatial frequencies

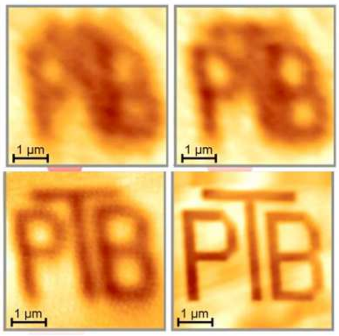
$z \gg \lambda$



Scanning Near-Field Optical Microscope (SNOM):

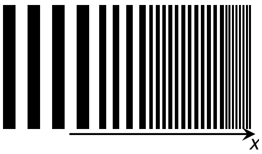


https://www.photonics.com/Articles/NSOM_Discovering_New_Worlds/a25127



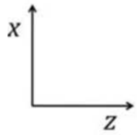
Change in resolution as a function of tip-sample distance

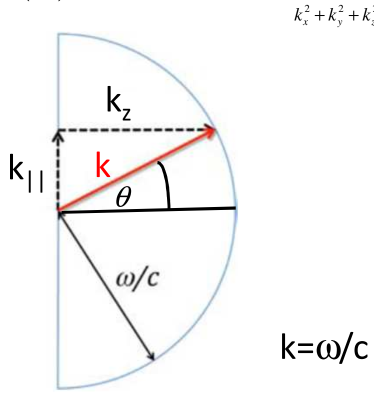
Propagation as a low-pass filter



$$E(x,y,z,\omega)=\frac{1}{(2\pi)^2}\iint dk_x dk_y E\left(k_x,k_y,z=0,\omega\right)e^{i\left(k_x x+k_y y+k_z z\right)}$$

$$k_x^2+k_y^2+k_z^2=\frac{\omega^2}{c^2}$$





$$k_z=\begin{cases}\sqrt{\dfrac{\omega^2}{c^2}-k_x^2-k_y^2} & \text{if } \dfrac{\omega^2}{c^2}>k_x^2+k_y^2\\ i\sqrt{k_x^2+k_y^2-\dfrac{\omega^2}{c^2}} & \text{otherwise}\end{cases}$$

Summary: resolution and spatial frequencies

Exact solution to our problem:

Knowing the distribution of the electric field on a plane at z=0, can we find an expression for the field at a distance z>0?

$$E(x,y,z,\omega)=\frac{1}{(2\pi)^2}\iint dk_x dk_y E(k_x,k_y,z=0,\omega)\, e^{i(k_x x+k_y y+k_z z)}$$

$$E(k_x,k_y,z=0,\omega)=\iint dxdy E(x,y,z=0,\omega)\, e^{-i(k_x x+k_y y)}$$

…but still a bit complicated to calculate!

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Fraunhofer or far-field diffraction

Two methods:

- starting with the Huygens-Fresnel principle of secondary wavelets
- using the stationary phase approximation

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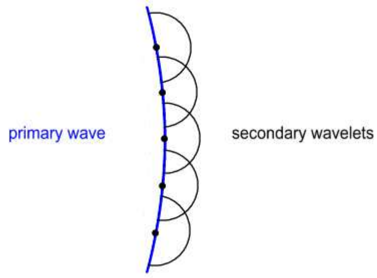
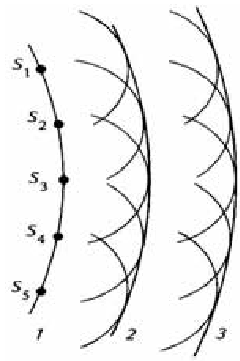
Huygens-Fresnel principle

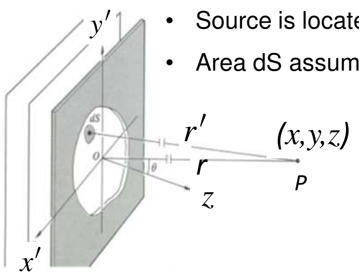
Every point on a primary wavefront serves as the source of spherical secondary wavelets such that the primary wavefront at some later time is the envelope of these wavelets. The wavelets advance with a speed and frequency equal to those of the primary wave.





http://www.fisica.uniud.it/irdis/Ottica/Diffrazione_guida/DiffrazioneGuida.htm

Fraunhofer approximation



- Source is located at z=0
- Area dS assumed covered with coherent point sources; dS << λ ;

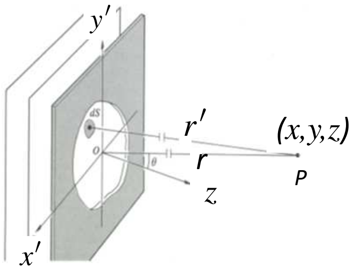
- dS assumed to emit a ***spherical wave***
- $\mathcal{E}_A$ is the source strength per unit area $= E(x', y', 0, \omega)$

$$dE \approx -\frac{i}{\lambda}\mathcal{E}_A \frac{e^{ikr'}}{r'} dS$$

Fraunhofer condition:

Point of observation at a distance >> size of source

Fraunhofer approximation



$$dE \approx -\frac{i}{\lambda}\mathcal{E}_A \frac{e^{ikr'}}{r'} dS$$

$$r' = \left[(x-x')^2 + (y-y')^2 + z^2\right]^{1/2}$$

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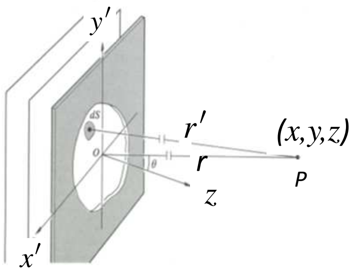
Fraunhofer approximation



$$dE \approx -\frac{i}{\lambda}\mathcal{E}_A \frac{e^{ikr'}}{r'} dS$$

$$r' \approx z\left(1 - \frac{xx' + yy'}{z^2}\right)$$

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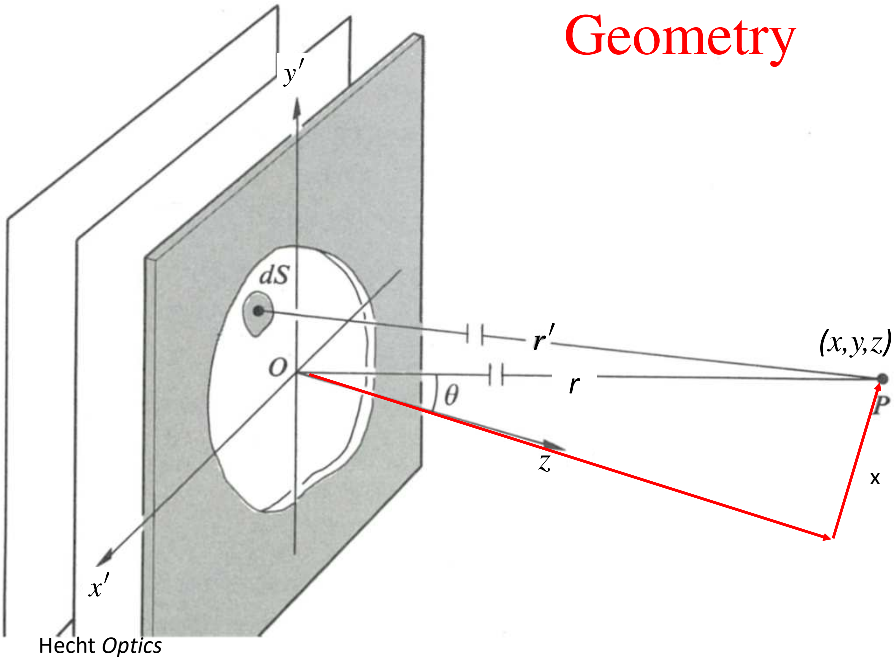
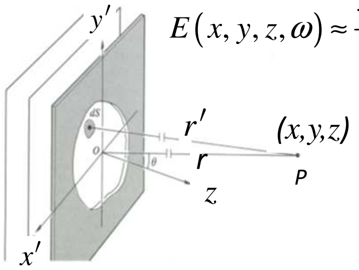
Geometry



Hecht *Optics*

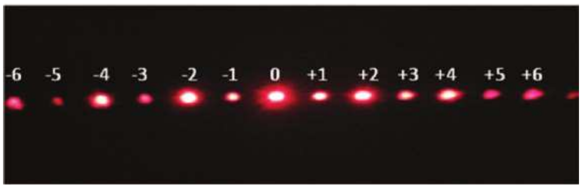
Fraunhofer approximation

$$E(x,y,z,\omega) \approx \frac{-i}{\lambda}\frac{e^{ikz}}{z}\iint_{aperture} dx'dy'\left[E(x',y',0,\omega)\,e^{-ik_x x' - ik_y y'}\right]$$



The field distribution in the Fraunhofer diffraction pattern is proportional to the Fourier transform of the field distribution across the aperture!!!

Far field pattern: 1-D periodic grating



$$\left|E\left(k_x, k_y, z=0, \omega\right)\right|^2$$

https://www.researchgate.net/figure/1D-diffraction-pattern-of-CLC-grating-probed-by-He-Ne-laser_fig3_254248651

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Method 2: Stationary phase approximation

$$E\left(x, y, z, \omega\right) = \frac{1}{(2\pi)^2}\iint dk_x dk_y E\left(k_x, k_y, z=0, \omega\right) e^{i\left(k_x x + k_y y + k_z z\right)}$$ z large

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} = k\sqrt{1 - \frac{k_x^2}{k^2} - \frac{k_y^2}{k^2}}$$

$$\simeq k\left(1 - \frac{k_x^2}{2k^2} - \frac{k_y^2}{2k^2}\right) \quad \text{if } k_x, k_y \ll k$$

$$e^{i\left(k_x x + k_y y + k_z z\right)} \simeq e^{ikz} \exp\left[i\left(k_x x - \frac{k_x^2 z}{2k}\right)\right] \exp\left[i\left(k_y y - \frac{k_y^2 z}{2k}\right)\right]$$

$$\int dk_x E\left(k_x, k_y, 0, \omega\right) \exp\left[i\left(k_x x - \frac{k_x^2 z}{2k}\right)\right] = ???$$

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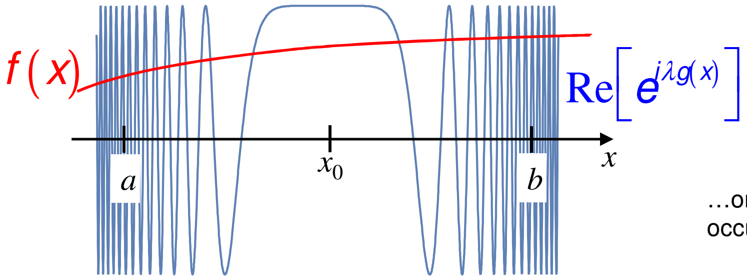
Stationary phase approximation

$$I(\lambda)=\int_a^b dx\, f(x)\, e^{i\lambda g(x)}$$

λ large

$$\int dk_x E(k_x,k_y,0,\omega)\exp\left[i\left(k_x x-\frac{k_x^2 z}{2k}\right)\right]$$



If $f(x)$: slowly varying amplitude

$g(x)$: rapidly varying phase

…only significant contribution to integral occurs at a stationary point, i.e. where

$$g'(x)=0$$

If $g'(x)$ has only one zero x_0 in the interval $[a,b]$:

$$I(\lambda)\simeq\sqrt{\frac{2\pi}{\lambda|g''(x_0)|}}f(x_0)e^{i\lambda g(x_0)}e^{i\frac{\pi}{4}\mathrm{sign}[g''(x_0)]}$$

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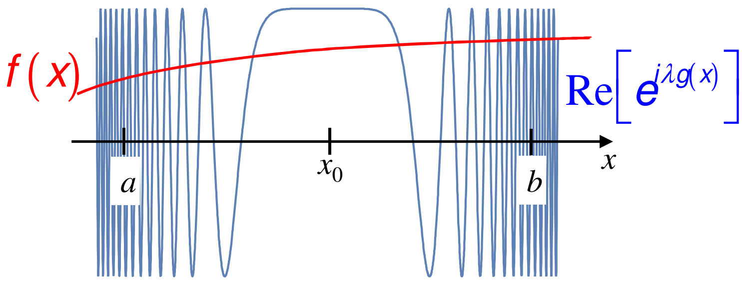
Stationary phase approximation

$$I(\lambda)=\int_a^b dx\, f(x)\, e^{i\lambda g(x)}$$

λ large

$$\int dk_x E(k_x,k_y,0,\omega)\exp\left[i\left(k_x x-\frac{k_x^2 z}{2k}\right)\right]$$



$f(x)$: slowly varying amplitude

$g(x)$: rapidly varying phase

$$\Rightarrow\quad g(k_x)=k_x x-\frac{k_x^2 z}{2k}$$

$$g'(k_x)=x-\frac{k_x z}{k}=0$$

$$\Rightarrow\quad k_x^0=\frac{kx}{z}\qquad g''(k_x^0)=-\frac{z}{k}$$

If $g'(x)$ has only one zero x_0 in the interval $[a,b]$:

$$I(\lambda)\simeq\sqrt{\frac{2\pi}{\lambda|g''(x_0)|}}f(x_0)e^{i\lambda g(x_0)}e^{i\frac{\pi}{4}\mathrm{sign}[g''(x_0)]}$$

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Method 2: Stationary phase approximation

z large

$$E(x,y,z,\omega)=\frac{1}{(2\pi)^2}\iint dk_x dk_y E(k_x,k_y,z=0,\omega)e^{i(k_x x+k_y y+k_z z)}$$

$$k_z=\sqrt{k^2-k_x^2-k_y^2}=k\sqrt{1-\frac{k_x^2}{k^2}-\frac{k_y^2}{k^2}}$$

$$\simeq k\left(1-\frac{k_x^2}{2k^2}-\frac{k_y^2}{2k^2}\right)\quad \text{if } k_x,k_y\ll k$$

$$e^{i(k_x x+k_y y+k_z z)}\simeq e^{ikz}\exp\left[i\left(k_x x-\frac{k_x^2 z}{2k}\right)\right]\exp\left[i\left(k_y y-\frac{k_y^2 z}{2k}\right)\right]$$

Stationary phase approximation

$$\int dk_x E(k_x,k_y,0,\omega)\exp\left[i\left(k_x x-\frac{k_x^2 z}{2k}\right)\right]\simeq\sqrt{\frac{2\pi k}{z}}E\left(k_x=\frac{kx}{z},k_y,0,\omega\right)e^{i\frac{kx^2}{2z}}e^{-i\frac{\pi}{4}}$$

- Do same thing with respect to k_y
- Put everything together

$$E(x,y,z,\omega)=-\frac{i}{\lambda}E\left(k_x=\frac{kx}{z},k_y=\frac{ky}{z},z=0,\omega\right)\frac{e^{ikz}}{z}$$

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Method 2: Stationary phase approximation

$$E(x,y,z,\omega)=\frac{1}{(2\pi)^2}\iint dk_x dk_y E(k_x,k_y,z=0,\omega)e^{i(k_x x+k_y y+k_z z)}$$

z large

If $k\gg k_x,k_y$:

$$E(x,y,z,\omega)=-\frac{i}{\lambda}E\left(k_x=\frac{kx}{z},k_y=\frac{ky}{z},z=0,\omega\right)\frac{e^{ikz}}{z}$$

Fourier transform as a function of t of the electric field

Fourier transform as a function of t, x and y of the electric field

Far field diffraction = 2D spatial Fourier transform of incident field!!!

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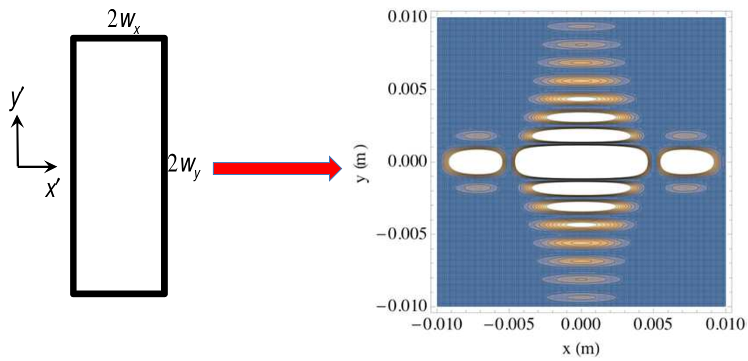
Fraunhofer diffraction: an example

$$E(x,y,z,\omega) = -\frac{i}{\lambda} E\left(k_x=\frac{kx}{z}, k_y=\frac{ky}{z}, z=0,\omega\right)\frac{e^{ikz}}{z}$$

$$\int_{-w_x}^{w_x} dx\, E_0 e^{-ik_x x} = 2E_0 \frac{\sin k_x w_x}{k_x}$$



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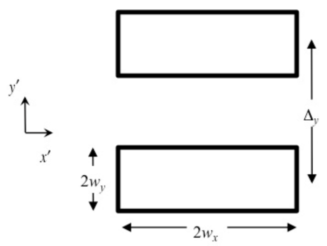
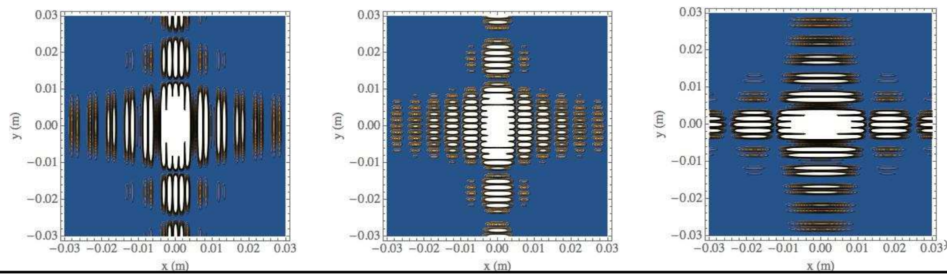
Fraunhofer diffraction: QUIZ

$$E(x,y,z,\omega) = -\frac{i}{\lambda} E\left(k_x=\frac{kx}{z}, k_y=\frac{ky}{z}, z=0,\omega\right)\frac{e^{ikz}}{z}$$





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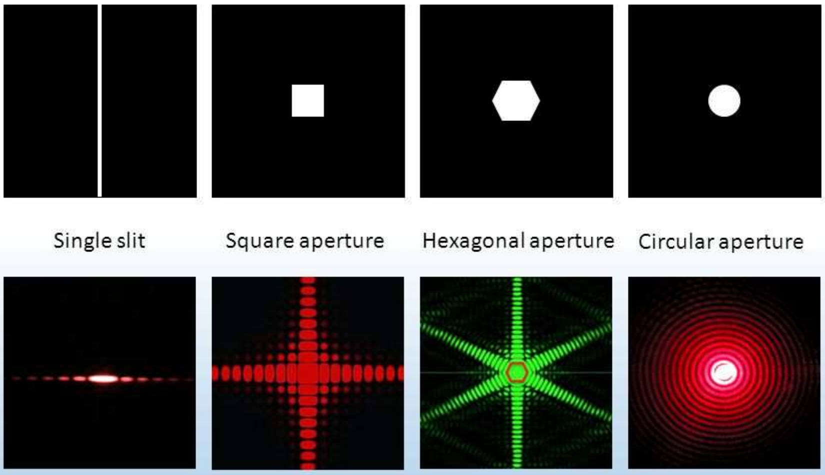
Diffraction in 2D



CheckPoint 2

Phys. 102, Lecture 23, Slide 10

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Summary

- Using Fourier analysis we can **express a wave as a sum of plane waves**
- Each wave in the sum corresponds to a **specific spatial frequency** and propagates **in a specific direction** with respect to the optical axis
- Spatial frequencies $|k_z| > \omega/c$ give rise to **evanescent waves which do not propagate**
- Diffraction and propagation is thus a type of **spatial filtering**
- In the far-field, **the resulting diffraction pattern is the spatial 2D Fourier transform of the intensity at z=0**

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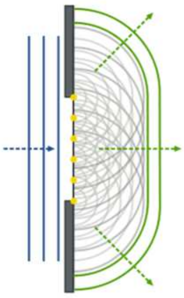
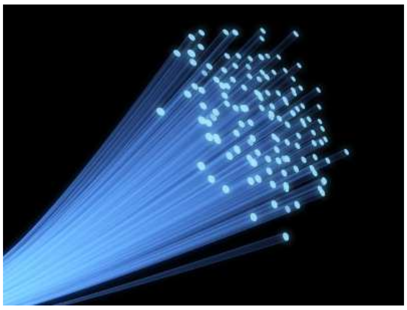
Lecture 5: (More) diffraction and waveguides

Goals today:

- Express an arbitrary wave as a sum of *spherical* waves
- Huygens-Fresnel principle
- Fresnel approximation—diffraction *before* the far-field
- Waveguides





https://en.wikipedia.org/wiki/Huygens%E2%80%93Fresnel_principle

https://qualitysurgicalrepairs.com/video_cameras__consoles___fiberoptic_cable

Summary: express an arbitrary wave as a sum of plane waves (same $|k|$, different $k_x, k_y, k_z(k_x, k_y)$)

$E(x, y, z=0, t)$

FT $t \to \omega$ — Decompose in terms of spectral components

$E(x, y, z=0, \omega)$

FT $x,y \to k_x,k_y$ — Decompose in terms of plane wave components

$E(k_x, k_y, z=0, \omega)$

$\times e^{jk_z z}$ Plane wave

k_z function of k_x, k_y, ω

$E(k_x, k_y, z, \omega) = E(k_x, k_y, z=0, \omega) \times e^{jk_z z}$

FT^{-1} $k_x,k_y \to x,y$

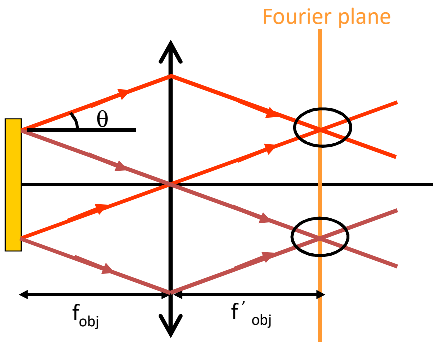
$E(x, y, z, \omega)$

(Directly from $E(x, y, z=0, \omega)$ to $E(x, y, z, \omega)$: Too hard!)

FT^{-1} $\omega \to t$

$E(x, y, z, t)$

Spatial frequences



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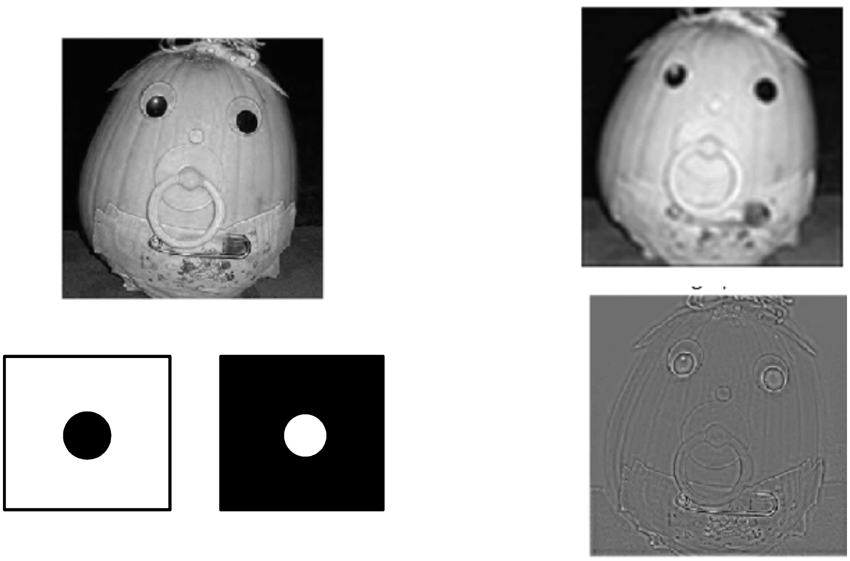
Spatial frequencies



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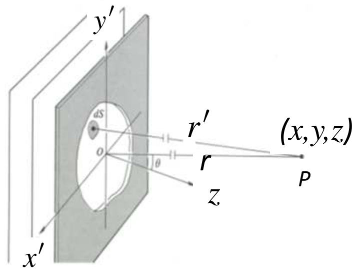
Fraunhofer approximation



Fraunhofer condition:

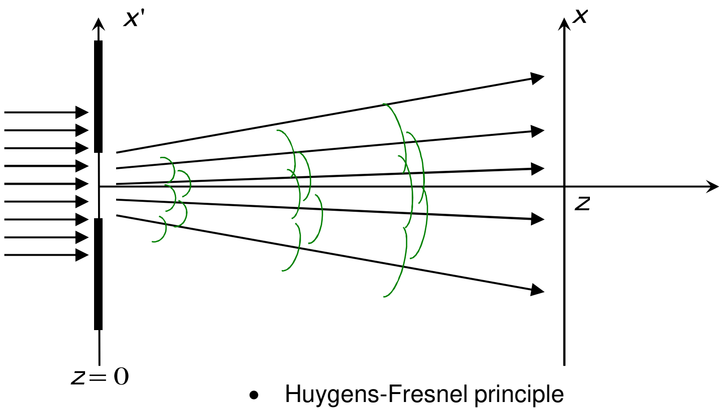
Point of observation at a distance >> size of source

$$E(x,y,z,\omega) = -\frac{i}{\lambda} E\left(k_x = \frac{kx}{z}, k_y = \frac{ky}{z}, z = 0, \omega\right)\frac{e^{ikz}}{z}$$

Fourier transform as a function of t, x and y of the electric field

Far field diffraction = 2D spatial Fourier transform of incident field!!!

Now: express field as a sum of *spherical* waves



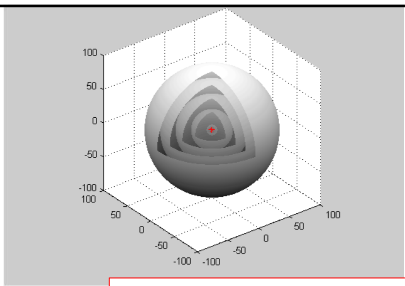
- Huygens-Fresnel principle
- Fresnel approximation—diffraction *before* the far-field

Arbitrary field as a sum of *spherical waves*: Rayleigh-Sommerfeld expression

Recall: Field as a sum of *plane* waves:

$$E(x,y,z,\omega)=\frac{1}{(2\pi)^2}\iint dk_x dk_y E(k_x,k_y,z=0,\omega)e^{ik_z z}e^{i(k_x x+k_y y)}$$

"The Fourier transform of a product is equal to the convolution of the separate Fourier transforms".



Weyl plane wave decomposition of a spherical wave

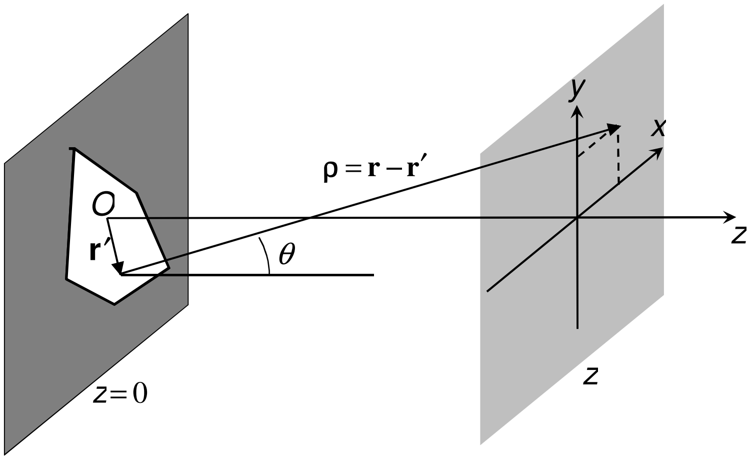
$$\frac{\exp(ikr)}{r}=\frac{i}{2\pi}\iint dk_x dk_y \frac{\exp\left[i\left(k_x x+k_y y+k_z|z|\right)\right]}{k_z} \quad (3.18)$$

$$E(x,y,z,\omega)=E(x,y,z=0,\omega)*FT^{-1}\left\{e^{ik_z z}\right\}$$

$$k_x^2+k_y^2+k_z^2=\frac{\omega^2}{c^2}$$

In the search to find $FT^{-1}\left\{e^{ik_z z}\right\}$ calculate $\frac{\partial}{\partial z}\frac{\exp(ikr)}{r}$

Geometry



$$\rho(\mathbf{r},\mathbf{r}') = \|\mathbf{r}-\mathbf{r}'\| = \sqrt{(x-x')^2+(y-y')^2+z^2}$$

Towards the Rayleigh-Sommerfield relation

$$E(x,y,z,\omega) = E(x,y,z=0,\omega) * \left(-\frac{1}{2\pi}\frac{\partial}{\partial z}\frac{\exp(ikr)}{r}\right)$$

$$h(\vec{x}) * g(\vec{x}) = \int_{-\infty}^{\infty} d\vec{x}'\, h(\vec{x}')\, g(\vec{x}-\vec{x}') \qquad \rho(\mathbf{r},\mathbf{r}') = \|\mathbf{r}-\mathbf{r}'\| = \sqrt{(x-x')^2+(y-y')^2+z^2}$$

Towards the Huygens-Fresnel principle

$$E(x,y,z,\omega) = -\frac{1}{2\pi}\iint dx'dy' E(x',y',z=0,\omega)\frac{\partial}{\partial z}\frac{\exp[ik\rho(\mathbf{r},\mathbf{r}')]}{\rho(\mathbf{r},\mathbf{r}')} \quad (3.20)$$

Rayleigh-Sommerfeld relation

Find: $\frac{\partial}{\partial z}\frac{e^{ik\rho}}{\rho}$ $\qquad \rho(\mathbf{r},\mathbf{r}') = \|\mathbf{r}-\mathbf{r}'\| = \sqrt{(x-x')^2+(y-y')^2+z^2}$

$$E(x,y,z,\omega) = -\frac{1}{2\pi}\iint dx'dy' E(x',y',z=0,\omega)\frac{\exp[ik\rho(\mathbf{r},\mathbf{r}')]}{\rho(\mathbf{r},\mathbf{r}')^2} z\left(-\frac{1}{\rho(\mathbf{r},\mathbf{r}')}+ik\right)$$

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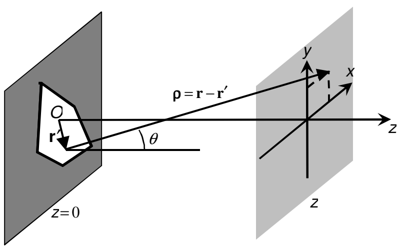
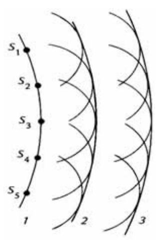
$\rho(\mathbf{r},\mathbf{r}') = \|\mathbf{r}-\mathbf{r}'\| = \sqrt{(x-x')^2+(y-y')^2+z^2}$

Huygens-Fresnel Principle

$$E(x,y,z,\omega) = -\frac{1}{2\pi}\iint dx'dy' E(x',y',z=0,\omega)\frac{\exp[ik\rho(\mathbf{r},\mathbf{r}')]}{\rho(\mathbf{r},\mathbf{r}')^2} z\left(-\frac{1}{\rho(\mathbf{r},\mathbf{r}')}+ik\right)$$





$$E(x,y,z,\omega) = -\frac{i}{\lambda}\iint dx'dy' E(x',y',z=0,\omega)\frac{\exp[ik\rho(\mathbf{r},\mathbf{r}')]}{\rho(\mathbf{r},\mathbf{r}')}\cos\theta \quad (3.24)$$

Huygens-Fresnel principle

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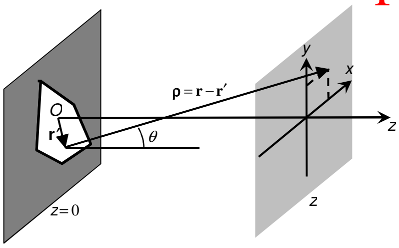
Fresnel approximation



- Valid "before the far field" (to be defined more precisely).

$$\rho(\mathbf{r},\mathbf{r}') = \|\mathbf{r}-\mathbf{r}'\| = \sqrt{(x-x')^2+(y-y')^2+z^2}$$

$$E(x,y,z,\omega) = -\frac{i}{\lambda}\iint dx'dy' E(x',y',z=0,\omega)\frac{\exp[ik\rho(\mathbf{r},\mathbf{r}')]}{\rho(\mathbf{r},\mathbf{r}')}\cos\theta \qquad (3.24)$$

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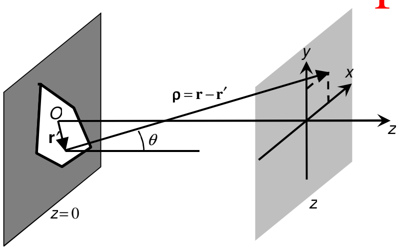
Fresnel approximation



$$\rho(\mathbf{r},\mathbf{r}') = \|\mathbf{r}-\mathbf{r}'\| = \sqrt{(x-x')^2+(y-y')^2+z^2}$$

$$\rho(\mathbf{r},\mathbf{r}') \simeq z\left[1+\frac{1}{2}\left(\frac{x-x'}{z}\right)^2+\frac{1}{2}\left(\frac{y-y'}{z}\right)^2\right]$$

$$E(x,y,z,\omega) = -\frac{i}{\lambda}\iint dx'dy' E(x',y',z=0,\omega)\frac{\exp[ik\rho(\mathbf{r},\mathbf{r}')]}{\rho(\mathbf{r},\mathbf{r}')}\cos\theta \qquad (3.24)$$

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Fresnel approximation: summary

$$E(x,y,z,\omega) = -\frac{ie^{ikz}}{\lambda z}\iint dx'dy' E(x',y',z=0,\omega)\exp\left\{i\frac{k}{2z}\left[(x-x')^2+(y-y')^2\right]\right\}$$

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Connection to Fraunhofer approximation

Starting with

$$E(x,y,z,\omega) = -\frac{ie^{ikz}}{\lambda z}\iint dx'dy' E(x',y',z=0,\omega)\exp\left\{i\frac{k}{2z}\left[(x-x')^2+(y-y')^2\right]\right\} \qquad (3.48)$$

$$E(x,y,z,\omega) = -\frac{i}{\lambda}E\left(k_x=\frac{kx}{z},k_y=\frac{ky}{z},z=0,\omega\right)\frac{e^{ikz}}{z}$$

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Link between Fresnel diffraction and the plane wave expansion

Fresnel Difraction:

$$E(x,y,z,\omega) = -\frac{ie^{ikz}}{\lambda z}\iint dx'dy' E(x',y',z=0,\omega)\exp\left\{i\frac{k}{2z}\left[(x-x')^2+(y-y')^2\right]\right\}$$

Fresnel Diffraction = convolution of the field at $z = 0$ with the transfer function

$$h_{\text{Fresnel}}(x,y,z,\omega) = -\frac{ie^{ikz}}{\lambda z}\exp\left[i\frac{k}{2z}\left(x^2+y^2\right)\right]$$

The FT of this transfer function is:

$$h_{\text{Fresnel}}(k_x,k_y,z,\omega) = e^{ikz}\exp\left[-i\frac{z}{2k}\left(k_x^2+k_y^2\right)\right]$$

$$F(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right) \longleftrightarrow F(\omega) = \sqrt{2\pi}\sigma\exp\left(-\frac{\omega^2\sigma^2}{2}\right)$$

$$E(k_x,k_y,z,\omega) = E(k_x,k_y,z=0,\omega)\bullet h_{Fresnel}(k_x,k_y,z,\omega)$$

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Link between Fresnel diffraction and the plane wave expansion

Fresnel: $E(k_x,k_y,z,\omega) = E(k_x,k_y,z=0,\omega)\bullet h_{Fresnel}(k_x,k_y,z,\omega)$ $\quad h_{\text{Fresnel}}(k_x,k_y,z,\omega) = e^{ikz}\exp\left[-i\frac{z}{2k}\left(k_x^2+k_y^2\right)\right]$

Plane wave expansion:

$$E(k_x,k_y,z,\omega) = E(k_x,k_y,z=0,\omega)\,e^{ik_z z}$$

Plane wave expansion = product of field at $z = 0$ and transfer function

$$h_{\text{plane\_waves}}(k_x,k_y,z,\omega) = \begin{cases} \exp\left[ikz\sqrt{1-\frac{k_x^2}{k^2}-\frac{k_y^2}{k^2}}\right] & \text{if } k_x^2+k_x^2<k^2 \\ 0 \quad \text{otherwise for } z \gg \lambda & \text{(evanescent waves)} \end{cases}$$

$$h_{\text{plane\_waves}}(k_x,k_y,z,\omega) \simeq e^{ikz}\exp\left[-i\frac{z}{2}\left(\frac{k_x^2}{k}+\frac{k_y^2}{k}\right)\right] = h_{\text{Fresnel}}(k_x,k_y,z,\omega)$$

Thus the Fresnel approximation is valid for k_x, $k_y << k$, i.e., for *small diffraction angles* => PARAXIAL APPROXIMATION

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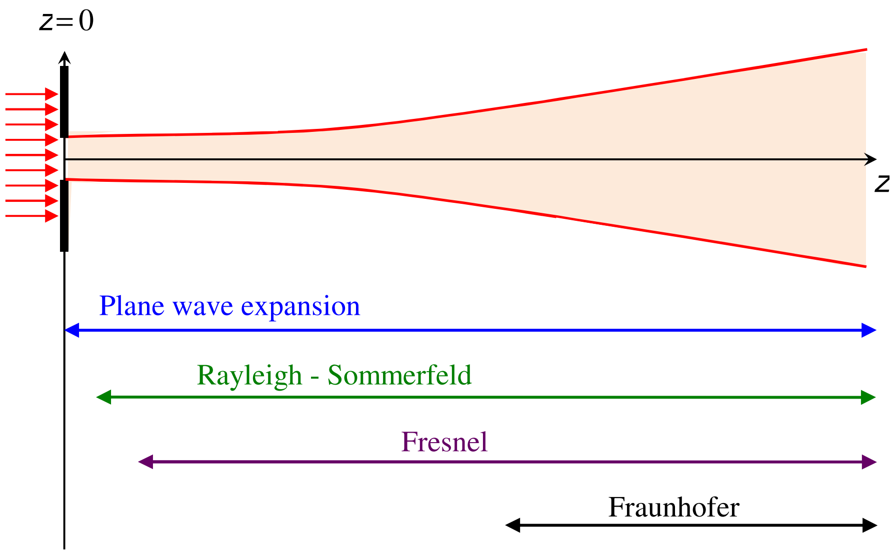
Conclusion: validity of the different formulations for diffraction



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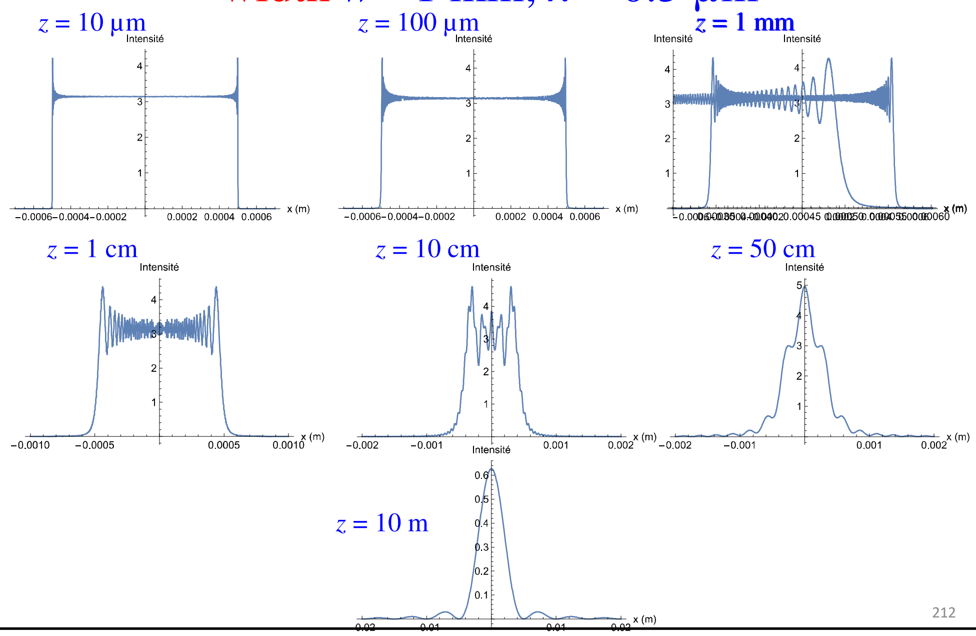
Example: Fresnel diffraction for a slit of width w =1 mm; λ = 0.5 µm



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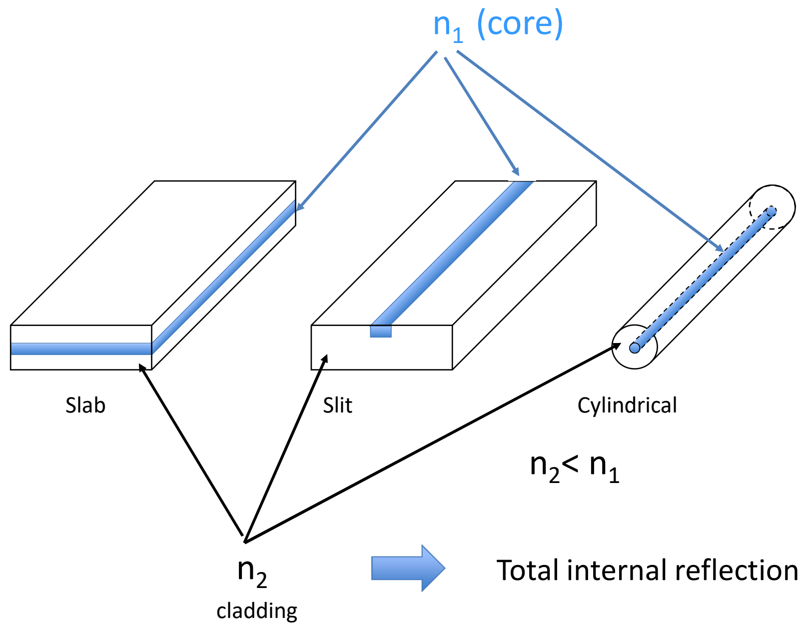
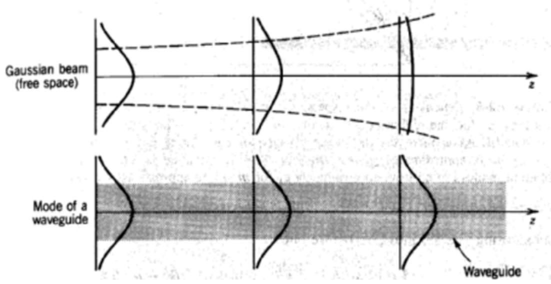
Waveguides



Why study waveguides?

Avoid diffraction



Why study waveguides?

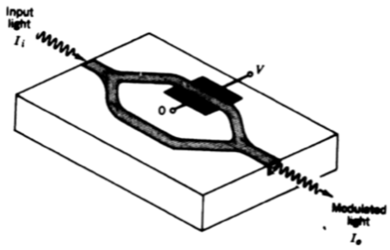
Electrical
response time

$$\tau = RC$$

$$C = \frac{\varepsilon A}{d}$$

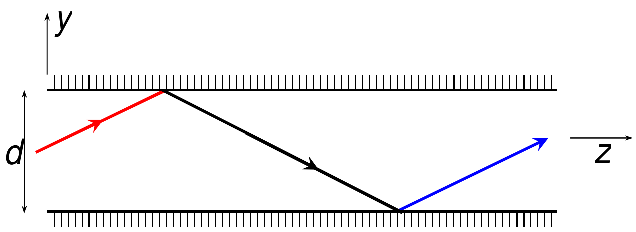
Towards integrated optoelectronics...

Small
=
Fast

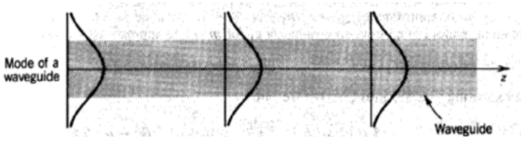


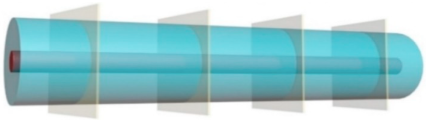
Metallic planar waveguide

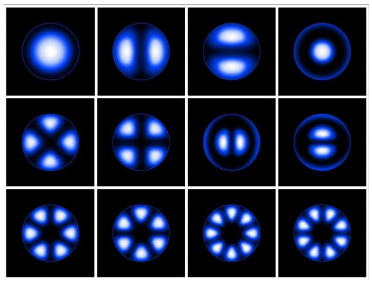
E field polarized in x direction



Optical modes







https://www.photonics.com/Articles/Large-Mode-Area_Optical_Fibers_Maintain/a62269

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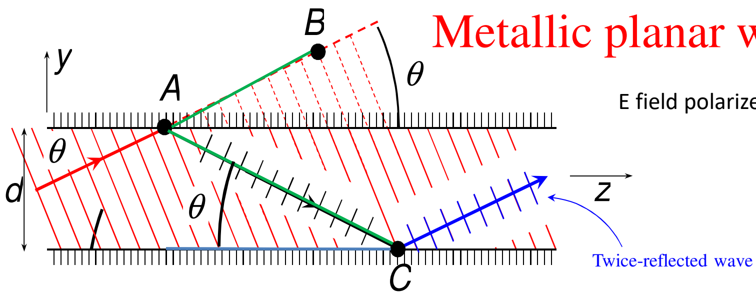
Metallic planar waveguide



E field polarized in x direction

Mode: Twice-reflected wave must be identical to the incident wave

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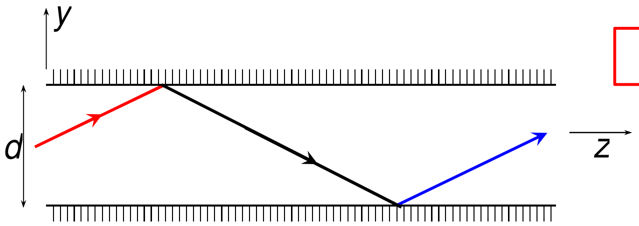
Fundamental mode, single mode waveguides



$\sin\theta_m = m\frac{\lambda}{2d}$ $m = 1, 2, 3....$

Single mode if $2d > \lambda > d$ ➡ Micron-sized waveguides

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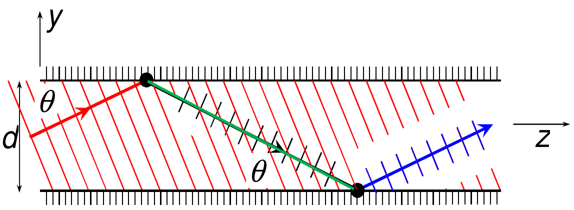
Propagation constants, group and phase velocities



$\sin\theta_m = m\frac{\lambda}{2d}$ $m = 1, 2, 3....$

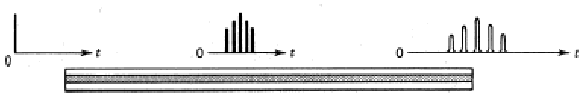
➡ $k_y^{(m)} = k\sin\theta_m$

$\beta_m = k_z^{(m)} = k\cos\theta_m$

$$v_\varphi^{(m)} = \frac{\omega}{\beta_m}$$

$$v_g^{(m)} = \frac{d\omega}{d\beta_m}$$

v_g depends on m:
intermodal
dispersion



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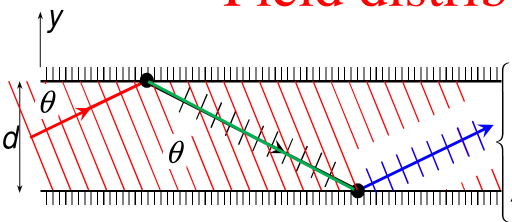
Field distributions



Mode of order m

$$k_y^{(m)} = k\sin\theta_m = m\frac{\pi}{d}$$

$$\beta_m = k_z^{(m)} = k\cos\theta_m = \sqrt{k^2 - m^2\frac{\pi^2}{d^2}}$$

Mode of order m: interference between plane wave with wave vectors $\left(0, k_y^{(m)}, \beta_m\right)$ and $\left(0, -k_y^{(m)}, \beta_m\right)$, in such a way that the fields cancel at the mirrors.

$$\mathbf{E}_m(\mathbf{r},t) = \mathcal{E}_m \hat{\mathbf{x}} u_m(y) \exp\left[-i(\omega t - \beta_m z)\right] + \text{c.c.}$$

$$u_m(y) = \begin{cases} \sqrt{\dfrac{2}{d}}\cos\dfrac{m\pi y}{d} & \text{for } m = 1,3,5\ldots \\ \sqrt{\dfrac{2}{d}}\sin\dfrac{m\pi y}{d} & \text{for } m = 2,4,6\ldots \end{cases}$$

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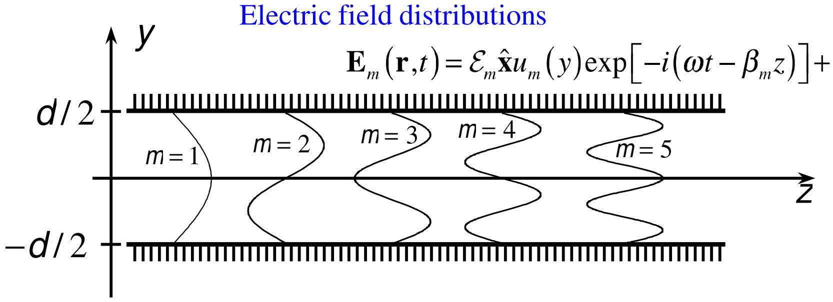
Field distributions

Electric field distributions

$$\mathbf{E}_m(\mathbf{r},t) = \mathcal{E}_m \hat{\mathbf{x}} u_m(y) \exp\left[-i(\omega t - \beta_m z)\right] + \text{c.c.}$$



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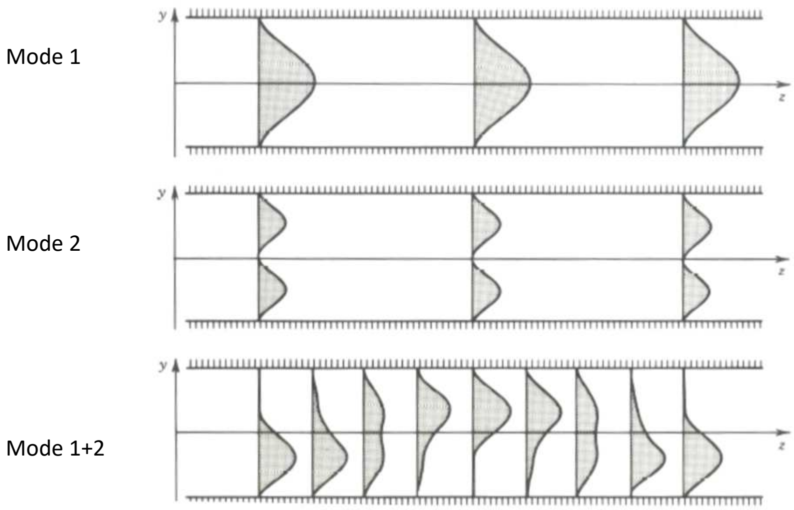
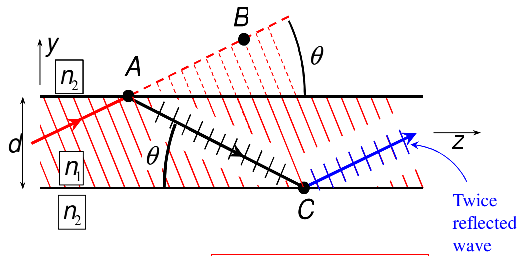
Metallic planar waveguide

Multimode fields



Saleh and Teich, *Fundamentals of Photonics, p. 247*

Planar dielectric waveguide



- Medium with index n_1 between two media with lower indices
- 2D problem: invariant along the x direction
- Propagation in the *yz* plane
- Electric field in the *x* direction

Total internal reflection: $\theta < \theta_c = \arccos \frac{n_2}{n_1}$

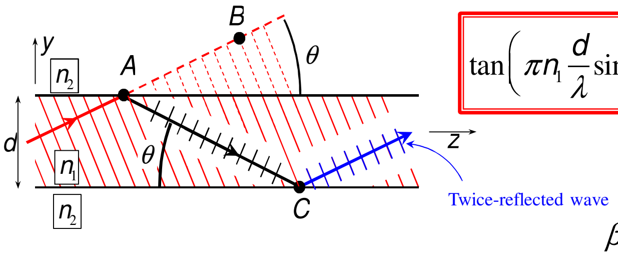
"Self-consistency condition"

with $\tan \frac{\varphi_r^{TE}}{2} = -\sqrt{\frac{\sin^2 \theta_c}{\sin^2 \theta} - 1}$

$$\frac{2\pi n_1}{\lambda} 2d \sin\theta + 2\varphi_r = 2m\pi$$

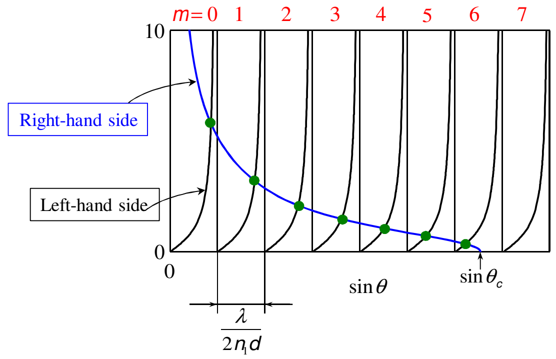
$$\tan\left(\pi n_1 \frac{d}{\lambda}\sin\theta - m\frac{\pi}{2}\right) = \sqrt{\frac{\sin^2\theta_c}{\sin^2\theta} - 1}$$

Planar dielectric waveguide



$$\tan\left(\pi n_1 \frac{d}{\lambda}\sin\theta - m\frac{\pi}{2}\right) = \sqrt{\frac{\sin^2\theta_c}{\sin^2\theta} - 1}$$

$$\beta_m = k_z^{(m)} = \frac{2\pi n_1}{\lambda}\cos\theta_m$$



$$\frac{2\pi n_2}{\lambda} \le \beta_m \le \frac{2\pi n_1}{\lambda}$$

There is always at least one mode.

Monomode if: $\frac{\lambda}{2n_1 d} > \sin\theta_c$

$$\frac{2d}{\lambda}\sqrt{n_1^2 - n_2^2} < 1$$

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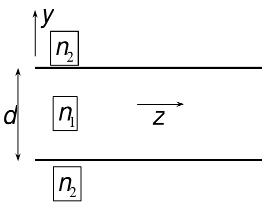
Planar dielectric waveguide

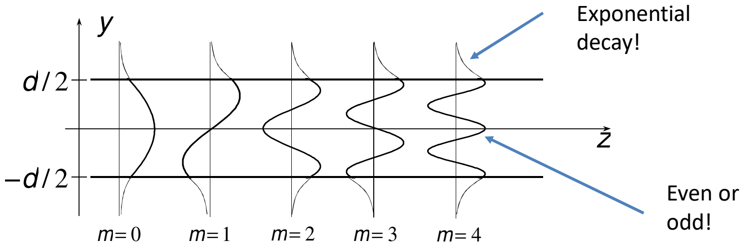


Propagation equation

$$\nabla^2 \mathbf{E}_m(\mathbf{r},t) - \frac{n^2(\mathbf{r})}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}_m(\mathbf{r},t) = 0$$

with $\mathbf{E}_m(\mathbf{r},t) = A\left[\mathcal{E}_m \hat{\mathbf{x}} u_m(y)\exp\left[-i(\omega t - \beta_m z)\right] + \text{c.c.}\right]$

$$\frac{d^2 u_m(y)}{dy^2} + \left[n^2(y)\frac{\omega^2}{c^2} - \beta_m^2\right] u_m(y) = 0$$



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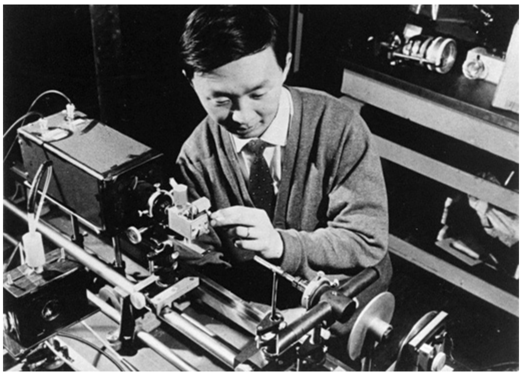
Nobel in Physics 2009 : Charles K. Kao

"for groundbreaking achievements concerning the transmission of light in fibers for optical communication"



http://nobelprize.org/nobel_prizes/physics/laureates/2009/index.html

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Conclusion

- Huygens-Fresnel = diffraction in terms of spherical waves
- Rayleigh-Sommerfeld, de Fresnel (= paraxial), and Fraunhofer approximations
- Metallic waveguides: based on reflection at metal surfaces. Problem of losses for non-ideal metals.
- Dielectric waveguides:
    - very low losses in the IR;
    - may be used to miniaturize opto-electronic components, thus increasing their bandwidth and decreasing their consumption;
    - along with the laser, are at the origin of the internet.

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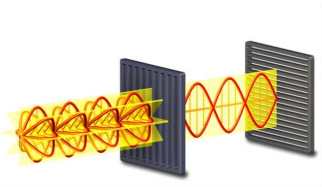
Lecture 6: Light propagation in ***anisotropic*** dielectric matter

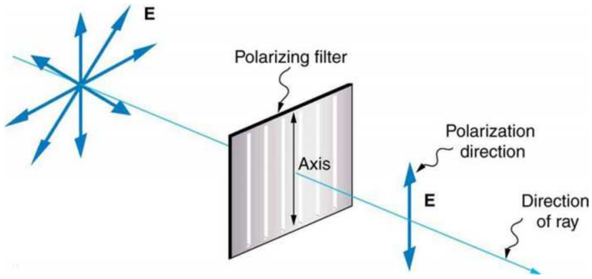
Anisotropy of materials

- Where does it come from?
- What are its consequences?

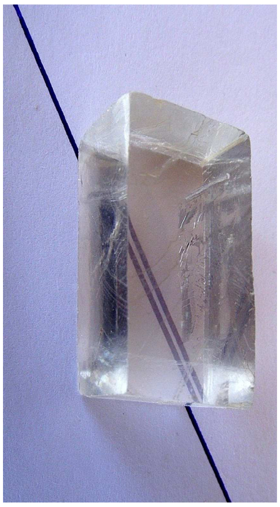


Why care about it?

Polarizers: control orientation of electric field of electromagnetic wave



Calcite

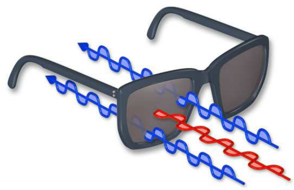


- Experiment:
  - Draw a black dot on a piece of paper
  - Place calcite crystal on top
  - Rotate crystal

  => observations
  - Look through polarizered sunglasses at crystal
  - Rotate crystal or sunglasses

  => observations

Applications



PHOTONICS MEDIA photonics.com

.: Diagnostic Method Employs Birefringence of Liquid Crystals

A detection system that uses birefringence as the sole optical output signal holds promise as a low-cost, rapid diagnostic tool for identifying biomarkers, viruses, bacteria and parasites in the field.

Scientists from ETH Zurich (the Swiss Federal Institute of Technology) used the phenomenon of birefringence of polarized light from lipid-based lyotropic liquid crystals, which consist of self-assembled structures of fat molecules in water.

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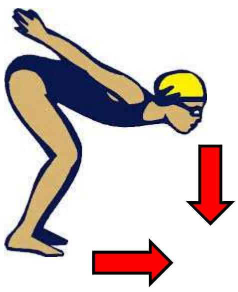
Anisotropy in daily life



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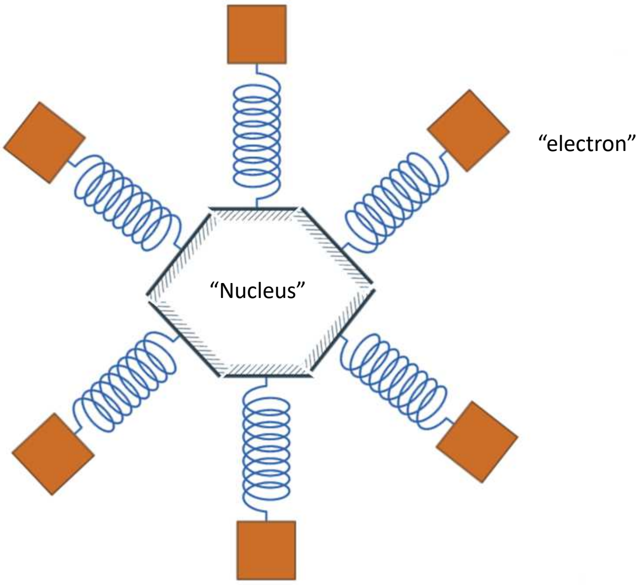
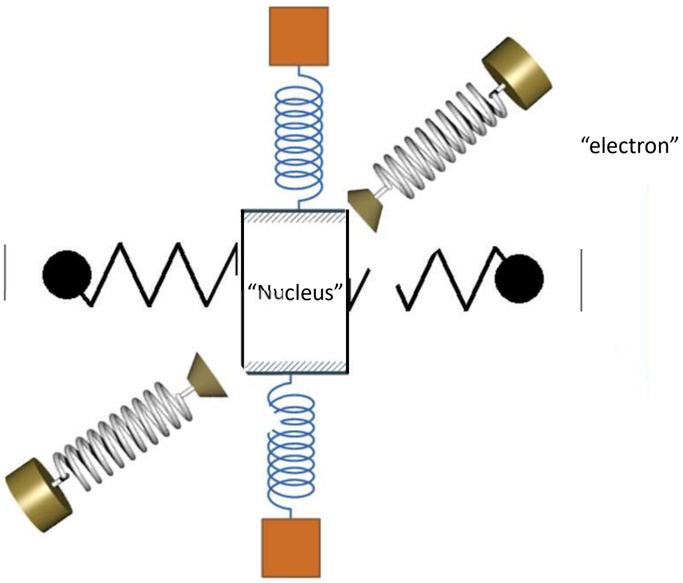
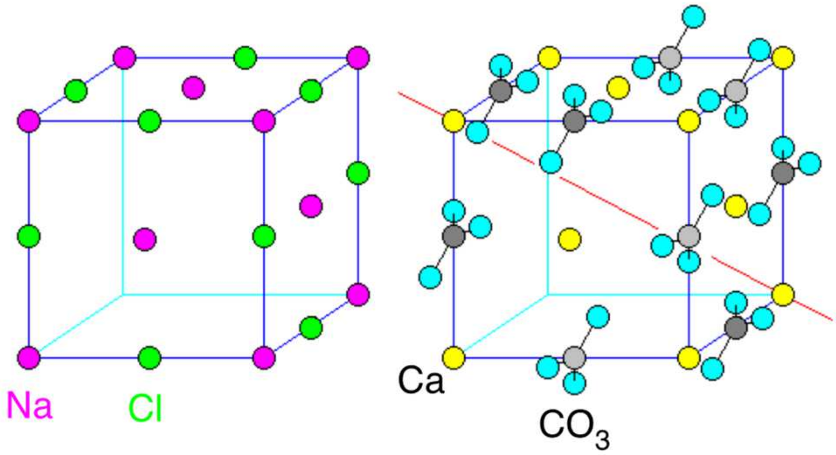
Recall: Light propagation in isotropic dielectric matter



Light propagation in ***anisotropic*** dielectric matter



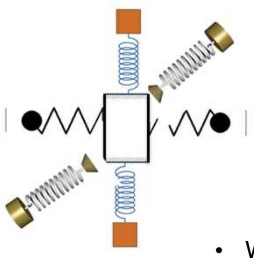
Anisotropic properties arise from an asymmetry in the atomic structure



Isotropic (cubic)　　Anisotropic

Consequences of anisotropy

- The index of refraction is now a ***tensor***



⇨ **D** and **E** are ***not*** in the same direction

⇨ The **speed of light** will be ***different*** depending on the **propagation direction** in the material.

- Wave vector **k** is (in general) **NOT** in the same direction as the **propagation of energy**

k: perpendicular to planes of a plane wave

Consequences of anisotropy

- A plane wave incident on an anisotropic crystal results (in general) in **two waves**:

  - one which acts in an **"ordinary"** (i.e. usual) fashion
  - one which acts in an **"extraordinary"** fashion *

* Not the general case

- The initial **polarization state** is (in general) **altered** for propagation in an anisotropic medium

  - There exists, however, **certain initial polarization directions** for which a linearly polarized light wave may travel **"unperturbed"**

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Medium characteristics

- homogeneous --same everywhere
- no losses --permittivity does NOT depend on frequency and is ***real***

Not like lecture 3!

- non-magnetic --$\mu = \mu_0$
- No free charges or currents…
- Linear and

$$\mathbf{D} = ???\,\mathbf{E}$$

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Dielectric permittivity tensor

Anisotropic matter: $\mathbf{D} = \varepsilon_0 \overleftrightarrow{\boldsymbol{\varepsilon}_\mathbf{r}} \mathbf{E}$

Tensor!

$$\mathbf{D} = \varepsilon_0 \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \quad \text{or} \quad D_k = \varepsilon_0 \sum_l \varepsilon_{kl} E_l; \quad k,l = 1,2,3$$

$$= \varepsilon_0 \begin{bmatrix} \varepsilon_{11}E_1 + \varepsilon_{12}E_2 + \varepsilon_{13}E_3 \\ \varepsilon_{21}E_1 + \varepsilon_{22}E_2 + \varepsilon_{23}E_3 \\ \varepsilon_{31}E_1 + \varepsilon_{32}E_2 + \varepsilon_{33}E_3 \end{bmatrix}$$

D and **E** are ***not*** in the same direction

$$\varepsilon_r = n^2$$

n: index of refraction

$$v_{\text{in medium}} = \frac{c}{n}$$

The **speed of light** will depend on the **propagation direction** in the material since $\overleftrightarrow{\boldsymbol{\varepsilon}_\mathbf{r}}$ is a tensor

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Dielectric permittivity tensor

"It may be shown" that in order to be consistent with the principle of conservation of energy ***the permittivity tensor must be symmetric***, **i.e.,**

$$\varepsilon_{kl} = \varepsilon_{lk} \qquad \mathbf{D} = \varepsilon_0 \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

Recall from math classes: a real symmetric matrix is ***diagonalizable.***

⇨ Can find eigenvalues and eigenvectors of this matrix!

⇨ The eigenvalues are the **principal dielectric constants**

⇨ The eigenvectors are the **principal dielectric axes**

⇨ Defined by the crystal structure of the material.

$$\boldsymbol{\varepsilon}_\mathbf{r} = \begin{pmatrix} n_1^2 & 0 & 0 \\ 0 & n_2^2 & 0 \\ 0 & 0 & n_3^2 \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon_1}{\varepsilon_0} & 0 & 0 \\ 0 & \frac{\varepsilon_2}{\varepsilon_0} & 0 \\ 0 & 0 & \frac{\varepsilon_3}{\varepsilon_0} \end{pmatrix}$$

Dielectric permittivity tensor in the principal axis coordinates

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Eigen phase velocities

Eigen phase, wave normal, or principal propagation velocities:

$$v_1 = c/n_1 \quad v_2 = c/n_2 \quad v_3 = c/n_3$$

$$\mathbf{u} = \mathbf{k}/k$$

Resulting phase velocities if the wave vector is perpendicular to one of the principal dielectric axes e.g., if **u**=(1,0,0), $v_\phi = v_2, v_3$

$$\mathbf{D} = \varepsilon_0 \begin{pmatrix} n_1^2 & 0 & 0 \\ 0 & n_2^2 & 0 \\ 0 & 0 & n_3^2 \end{pmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$D_i = \varepsilon_0 n_i^2 E_i = \varepsilon_0 \frac{c^2}{v_i^2} E_i = \frac{1}{\mu_0 v_i^2} E_i$$

D and **E** will not be parallel unless **E** is parallel to a principle axis, e.g., $\mathbf{E} = E_1 \hat{\mathbf{i}}$

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Some definitions

$$\mathbf{D} = \varepsilon_0 \begin{pmatrix} n_1^2 & 0 & 0 \\ 0 & n_2^2 & 0 \\ 0 & 0 & n_3^2 \end{pmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

$n_1 = n_2 = n_3$: matter is isotropic

$n_1 \neq n_2 = n_3$: crystal is uniaxial (calcite, quartz, ice)

$n_1 \neq n_2 \neq n_3$: crystal is biaxial (mica, topaz, borax)

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Plane wave propagation in an anisotropic medium

$$\mathbf{E} = \mathbf{E}_0 \exp\left[i\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right] \quad \text{(A)}$$

+ similar expressions for **D**, **B** and **H**.

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(a)} \qquad \nabla\cdot\mathbf{D} = 0 \quad \text{(b)}$$

$$\nabla\times\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \quad \text{(c)} \qquad \nabla\cdot\mathbf{B} = 0 \quad \text{(d)}$$

$$\mathbf{D} = \varepsilon_0 \overleftrightarrow{\boldsymbol{\varepsilon}}_{\mathbf{r}} \mathbf{E} \quad \text{(e)}$$

$$\mathbf{B} = \mu_0 \mathbf{H} \quad \text{(f)}$$

$$\nabla\times\left(f\mathbf{A}\right) = f\nabla\times\mathbf{A} + \nabla f\times\mathbf{A} \quad \text{(g)} \qquad \nabla\cdot\left(f\mathbf{A}\right) = f\nabla\cdot\mathbf{A} + \nabla f\cdot\mathbf{A} \quad \text{(h)}$$

From (a):

$$i\mathbf{k}\times\mathbf{E} = i\omega\mathbf{B} \quad (1)$$

From (b):

$$\mathbf{k}\cdot\mathbf{D} = 0$$

From (c), (f):

$$i\mathbf{k}\times\mathbf{B} = -i\omega\mu_0\mathbf{D} \quad (3)$$

From (d):

$$\mathbf{k}\cdot\mathbf{B} = 0 \quad (4)$$

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Plane wave in an anisotropic medium

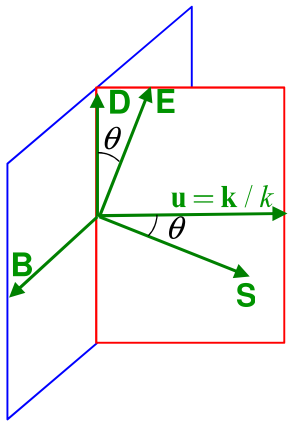
$$i\mathbf{k}\times\mathbf{E} = i\omega\mathbf{B} \quad (1) \qquad \mathbf{k}\cdot\mathbf{D} = 0 \neq \mathbf{k}\cdot\mathbf{E} \quad (2)$$

$$i\mathbf{k}\times\mathbf{B} = -i\omega\mu_0\mathbf{D} \quad (3) \qquad \mathbf{k}\cdot\mathbf{B} = 0 \quad (4) \qquad \text{(1.4)}$$

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \mathbf{E}\times\frac{\mathbf{B}}{\mu_0}$$

$(\mathbf{D}, \mathbf{B}) \perp \mathbf{k}$

$(\mathbf{E}, \mathbf{B}) \perp \mathbf{S}$



E, k (or **u**), **D, S** are coplanar

S: Poynting vector, direction of energy flow or rays

k: wave normal, direction perpendicular to wave fronts

S is NOT in the same direction as k !!

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For a given wave normal of direction **u**, what is the resulting phase velocity?

In other words, what is the index of refraction for this direction?

Let $\mathbf{u} = \frac{\mathbf{k}}{k}$ i.e., a unit vector in the direction of the wavevector

$\mathbf{k} = n\frac{\omega}{c}\mathbf{u}$ where n is the index of refraction for the direction **u**

Desired phase velocity: $$v_\varphi(\mathbf{u}) = \frac{\omega}{k} = \frac{c}{n}$$

First: find an expression for **D** in terms of **u**, **E** and v_ϕ

$$v_\varphi(\mathbf{u}) = \frac{\omega}{k} = \frac{c}{n}$$

$$i\mathbf{k}\times\mathbf{E} = i\omega\mathbf{B} \quad (1) \qquad \mathbf{k}\cdot\mathbf{D} = 0 \neq \mathbf{k}\cdot\mathbf{E} \quad (2)$$
$$i\mathbf{k}\times\mathbf{B} = -i\omega\mu_0\mathbf{D} \quad (3) \qquad \mathbf{k}\cdot\mathbf{B} = 0 \quad (4)$$

(1.4)

- From (3)

$$\mathbf{D} = -\frac{\mathbf{k}\times\mathbf{B}}{\omega\mu_0} = -\frac{k}{\omega}\frac{\mathbf{u}\times\mathbf{B}}{\mu_0} = -\frac{1}{v_\varphi}\frac{\mathbf{u}\times\mathbf{B}}{\mu_0} \quad (5)$$

- From (1)

$$\mathbf{B} = \frac{1}{v_\varphi}\frac{\mathbf{u}\times\mathbf{E}}{\mu_0} \quad (6)$$

- Plug (6) in (5)

$$\mathbf{D} = -\frac{1}{\mu_0 v_\varphi^2}\,\mathbf{u}\times(\mathbf{u}\times\mathbf{E})$$

- Use vector identity $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$$\mathbf{D} = \frac{1}{\mu_0 v_\varphi^2}\left[\mathbf{E} - \mathbf{u}(\mathbf{u}\cdot\mathbf{E})\right]$$

Plane wave in an anisotropic medium: find indices of refraction for a given propagation direction **u**

$$\mathbf{D} = \frac{1}{\mu_0 v_\varphi^2}\left[\mathbf{E} - \mathbf{u}(\mathbf{u}\bullet\mathbf{E})\right] \qquad (1.5)$$

$$D_i = \varepsilon_0 n_i^2 E_i = \varepsilon_0 \frac{c^2}{v_i^2} E_i = \frac{1}{\mu_0 v_i^2} E_i \qquad (1.3)$$

$$v_i = c/n_i \qquad \mathbf{D} = \varepsilon_0 \begin{pmatrix} n_1^2 & 0 & 0 \\ 0 & n_2^2 & 0 \\ 0 & 0 & n_3^2 \end{pmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \qquad \mathbf{u} = (u_1, u_2, u_3) = \frac{\mathbf{k}}{k}$$

- (1.3) = (1.5) ; solve for E_i

$$\Rightarrow \quad E_i = \frac{1/v_\varphi^2}{1/v_\varphi^2 - 1/v_i^2}(\mathbf{u}\cdot\mathbf{E})u_i \qquad (1.6)$$

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Plane wave in an anisotropic medium: find indices of refraction for a given propagation direction **u**

$$E_i = \frac{1/v_\varphi^2}{1/v_\varphi^2 - 1/v_i^2}(\mathbf{u}\cdot\mathbf{E})u_i \qquad (1.6)$$

- Multiply both sides by u_i

$$E_i u_i = \frac{1/v_\varphi^2}{1/v_\varphi^2 - 1/v_i^2}(\mathbf{u}\cdot\mathbf{E})u_i^2$$

- Add the resulting three equations (i=1,2,3)

$$\sum_{i=1,2,3} E_i u_i = \sum_{i=1,2,3} \frac{1/v_\varphi^2}{1/v_\varphi^2 - 1/v_i^2}(\mathbf{u}\cdot\mathbf{E})u_i^2$$

- Divide by $\mathbf{u}\cdot\mathbf{E}$

$$1 = \sum_{i=1,2,3} \frac{1/v_\varphi^2}{1/v_\varphi^2 - 1/v_i^2} u_i^2$$

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Plane wave in an anisotropic medium: find indices of refraction for a given propagation direction **u**

$$1 = \sum_{i=1,2,3} \frac{1/v_\varphi^2}{1/v_\varphi^2 - 1/v_i^2} u_i^2$$

•Subtract $1 = \sum_{i=1,2,3} u_i^2$

$$0 = \sum_{i=1,2,3} \left(\frac{1/v_\varphi^2}{1/v_\varphi^2 - 1/v_i^2} - 1 \right) u_i^2$$

•Simplify

$$\frac{u_1^2}{v_\varphi^2 - v_1^2} + \frac{u_2^2}{v_\varphi^2 - v_2^2} + \frac{u_3^2}{v_\varphi^2 - v_3^2} = 0 \qquad (1.7)$$

FRESNEL EQUATION

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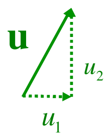
Fresnel equation



$$\frac{u_1^2}{v_\varphi^2 - v_1^2} + \frac{u_2^2}{v_\varphi^2 - v_2^2} + \frac{u_3^2}{v_\varphi^2 - v_3^2} = 0 \qquad (1.7)$$

$$v_\varphi(\mathbf{u}) = \frac{\omega}{k} = \frac{c}{n}$$

$$\left(v_\varphi^2 - v_2^2\right)\left(v_\varphi^2 - v_3^2\right)u_1^2 + \left(v_\varphi^2 - v_1^2\right)\left(v_\varphi^2 - v_3^2\right)u_2^2 + \left(v_\varphi^2 - v_1^2\right)\left(v_\varphi^2 - v_2^2\right)u_3^2 = 0$$

- Equation is of order 4 in v_φ and of order 2 in v_φ^2

➡ Get TWO solutions for v_φ !!!!

(Negative roots: waves propagating in the opposite direction!)

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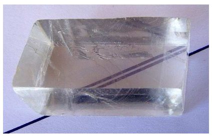
Fresnel equation: interpretation

- A plane wave in direction **u** will (in general) split into **two waves**, each with its own phase velocity!!!
- What determines how the plane wave splits? The **initial polarization**!
- These two waves propagate independently => will become **more and more** out of phase with each other as propagation continues
- Their wave vectors are in the same direction (**u**)…* **but *not* their Poynting vectors**!!!

The light associated with these waves will become separated with propagation!



* If no refraction

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What are the associated displacement vectors $\mathbf{d}'$ and $\mathbf{d}''$ for v'_φ and v''_φ?

Method 1: Algebraic method

- Use Fresnel equation to solve for v'_φ and v''_φ
- Use equation (1.6) to solve for electric field components:

$$E_i = \frac{1/v_\varphi^2}{1/v_\varphi^2 - 1/v_i^2}\left(\mathbf{u}\cdot\mathbf{E}\right)u_i \quad (1.6)$$

- Find ratios for $E_1 : E_2 : E_3$ and thus $D_1 : D_2 : D_3$
- Note: since everything is real in above equations, above ratios will be real, thus the **E** and **D** fields will be ***linearly polarized***

The structure of an anisotropic medium permits two monochromatic plane waves with two different linear polarizations $\mathbf{d}'$ and $\mathbf{d}''$ ***and two different velocities to propagate in a given direction***

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What if the initial wave polarization is along $\mathbf{d}'$ or $\mathbf{d}''$?

- The wave will NOT split into two
- The wave will travel at v'_φ or v''_φ
- The will be NO change in polarization
- Only one beam exists!

Finding v'_φ, v''_φ, $\mathbf{d}'$, and $\mathbf{d}''$ *graphically*

Recall: $v_\varphi(\mathbf{k}) = \frac{\omega}{k} = \frac{c}{n}$ $\quad \mathbf{u} = \frac{\mathbf{k}}{k}$

For a given propagation direction $\mathbf{u}$, what are the associated indices of refraction n' and n'', and displacement vectors $\mathbf{d}'$ and $\mathbf{d}''$

➡ index ellipsoid

Recall: $$\mathbf{D} = \frac{1}{\mu_0 \left[v_\varphi(\mathbf{k})\right]^2}\left[\mathbf{E} - (\mathbf{u}\cdot\mathbf{E})\mathbf{u}\right] \quad (1.5)$$

- Take dot product of each side with **D**; Recall: **D.u** = 0

➡ $$\mathbf{D}\cdot\mathbf{D} = \frac{1}{\mu_0 \left[v_\varphi(\mathbf{k})\right]^2}\mathbf{D}\cdot\mathbf{E}$$

Index ellipsoid

$$\mathbf{D}\cdot\mathbf{D}=\frac{1}{\mu_0\left[v_\varphi(\mathbf{k})\right]^2}\mathbf{D}\cdot\mathbf{E} \quad \text{(A)} \qquad v_\varphi(\mathbf{u})=\frac{\omega}{k}=\frac{c}{n} \quad \text{(B)}$$

Recall: $\mathbf{D}=\varepsilon_0\overleftrightarrow{\varepsilon}_\mathbf{r}\mathbf{E}$

Choose the coordinate system such that $\overleftrightarrow{\varepsilon}_\mathbf{r}$ is diagonal ➡ principal axes

➡ $D_i=\varepsilon_0 n_i^2 E_i$ (C)

Plug (C) and (B) into (A)

⬇

$$\mathbf{D}\cdot\mathbf{D}=n^2\left(\frac{D_1^2}{n_1^2}+\frac{D_2^2}{n_2^2}+\frac{D_3^2}{n_3^2}\right)$$

Define the following coordinate system:

$$(x_1,x_2,x_3)\equiv n\frac{\mathbf{D}}{\|\mathbf{D}\|}\equiv n\mathbf{d} \quad \text{or} \quad x_i\equiv n\frac{D_i}{\|\mathbf{D}\|}$$

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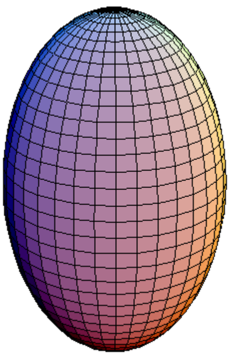
Index ellipsoid

$x_i\equiv n\frac{D_i}{\|\mathbf{D}\|}$ Solve for D_i, plug result into $\mathbf{D}\cdot\mathbf{D}=n^2\left(\frac{D_1^2}{n_1^2}+\frac{D_2^2}{n_2^2}+\frac{D_3^2}{n_3^2}\right)$

⬇

$$1=\frac{x_1^2}{n_1^2}+\frac{x_2^2}{n_2^2}+\frac{x_3^2}{n_3^2}$$

Index ellipsoid



- Set of all possible indices (for a crystal with n_1 n_2 n_3) consistent with conservation of energy
- Semi-axes of this ellipsoid are equal to n_1, n_2, n_3, the principal indices of refraction
- If $n_1=n_2=n_3$ Sphere! Isotropic!

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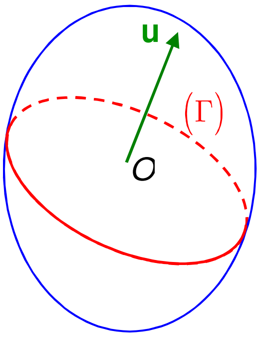
Finding the indices and displacement vectors ($\mathbf{d}''$ and $\mathbf{d}'$) for a given incident wave with wave normal $\mathbf{u}$

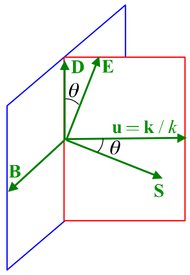
Index ellipsoid



- Propagation direction **u**, normal to wave fronts, through origin
- The intersection of a plane perpendicular to **u,** passing through the ellipsoid origin
  ➡ (in general) ellipse, Γ

Recall: $\mathbf{D} \perp \mathbf{u}$

$$(x_1, x_2, x_3) \equiv n\frac{\mathbf{D}}{\|\mathbf{D}\|} \equiv n\mathbf{d}$$



$n'\mathbf{d}'$, $n''\mathbf{d}''$, must be on this ellipse since $\mathbf{d}'$, $\mathbf{d}''$ are perpendicular to **u**

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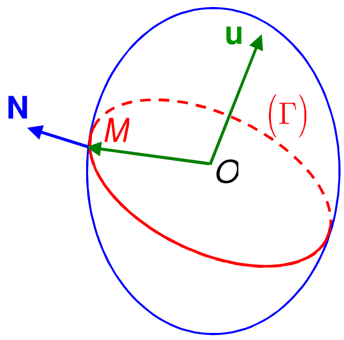
Finding the indices and displacement vectors ($\mathbf{d}'$, and $\mathbf{d}''$) for a given incident wave with wave normal $\mathbf{u}$



- Let OM be in the direction $\mathbf{d}'$ or $\mathbf{d}''$
- What is the ***normal*** to the ellipsoid at point M?
- From math, if $g(x_1, x_2, x_3) = \frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} - 1 = 0$

$$\mathbf{N} = \begin{pmatrix} \partial g / \partial x_1 \\ \partial g / \partial x_2 \\ \partial g / \partial x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 / n_1^2 \\ 2x_2 / n_2^2 \\ 2x_3 / n_3^2 \end{pmatrix}$$ is the normal

Thus

$$\mathbf{N} = \frac{2n}{\|\mathbf{D}\|}\begin{pmatrix} \frac{D_1}{n_1^2} \\ \frac{D_2}{n_2^2} \\ \frac{D_3}{n_3^2} \end{pmatrix} = \frac{2n\varepsilon_0}{\|\mathbf{D}\|}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \Rightarrow \mathbf{N} \parallel \mathbf{E}$$

$$x_i \equiv n\frac{D_i}{\|\mathbf{D}\|} \qquad D_i = \varepsilon_0 n_i^2 E_i$$

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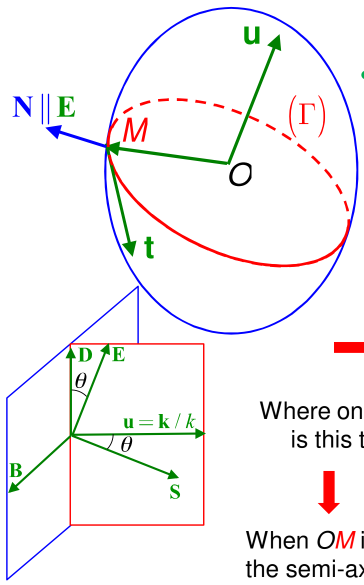
Finding the indices and displacement vectors for a given incident wave with wave normal $\mathbf{u}$



- **t:** tangent to Γ at M; in the plane of the ellipse Γ

➡ Thus $\mathbf{t} \perp \mathbf{u}$

- By definition $\mathbf{t} \perp \mathbf{N}$ and thus $\mathbf{t} \perp \mathbf{E}$

Recall:

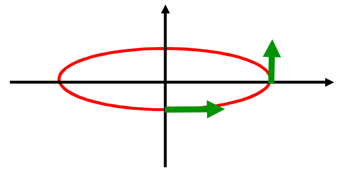
- **E** and **u** are in the same plane

➡ Thus $\mathbf{t} \parallel \mathbf{B}$ ➡ $\mathbf{t} \perp \mathbf{D}$ ➡ $\mathbf{t} \perp OM$

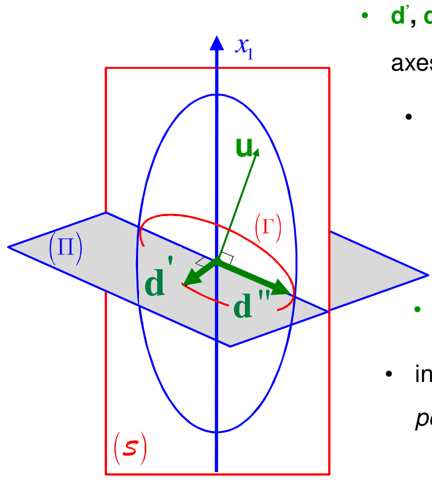
Where on an ellipse is this true???

⬇

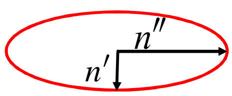
When *OM* is one of the semi-axes!!!



Graphical method for determining the directions of the two displacement vectors ($\mathbf{d}'$, and $\mathbf{d}''$) for a given incident wave with wave normal $\mathbf{u}$



- **d', d"** are in the ***directions*** of the two semi-axes of the ellipse Γ
  - the associated indices ***n', n"*** are the ***lengths*** of the semi-axes



- **d', d"** are ***perpendicular to each other***

⬇

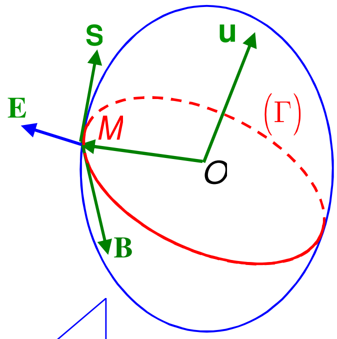
- incoming light splits into two *orthogonal linear polarizations*, each with its own phase velocity

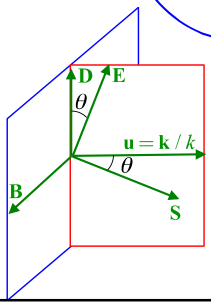
⬇

- initial polarization along **d'** or **d"**

➡ • one index, one velocity

Graphical method for determining the direction of the Poynting vector for a given incident wave with wave normal **u**





Recall:

- **S** is perpendicular to **B** and **E**
- **B** is tangential to the ellipsoid and **E** is normal

⬇

<u>**S** must be tangential to the ellipsoid in order to form the necessary orthogonal triad</u>

More handy geometrical tools: normal surfaces and optic axes

What is a ***<u>normal surface?</u>***

- Place an origin inside the crystal.
- Consider a wave vector direction **u**
- For each direction **u,** associate <u>two vectors</u> whose lengths are proportional to the <u>two corresponding indices of refraction *n'* and *n"*</u>
- Repeat for all values of **u.** The endpoints of the index vectors give rise to a surface consisting of two *"shells"* and known as ***<u>the wave-normal or normal surface</u>***
  - i.e., the set of all points N such that **<u>ON</u>**<u>=*n***u**</u>

Normal surfaces

Wave-normal or normal surface: the set of all points N such that **ON**=n**u**

- Start with Fresnel's Equation $$\frac{u_1^2}{v_\varphi^2 - v_1^2} + \frac{u_2^2}{v_\varphi^2 - v_2^2} + \frac{u_3^2}{v_\varphi^2 - v_3^2} = 0$$
- Let **ON** = (x,y,z) ➡ $$n^2 = x^2 + y^2 + z^2$$ $$u_1 = \frac{x}{n}, \quad u_2 = \frac{y}{n}, \quad u_3 = \frac{z}{n}$$
- Recall: $v_\varphi = \frac{c}{n}$ $\quad v_i = \frac{c}{n_i}$
- Plug the above expressions into the Fresnel equation to obtain:

$$\begin{aligned} &n_1^2 x^2 \left(x^2 + y^2 + z^2 - n_2^2\right)\left(x^2 + y^2 + z^2 - n_3^2\right) \\ &+ n_2^2 y^2 \left(x^2 + y^2 + z^2 - n_1^2\right)\left(x^2 + y^2 + z^2 - n_3^2\right) \\ &+ n_3^2 z^2 \left(x^2 + y^2 + z^2 - n_1^2\right)\left(x^2 + y^2 + z^2 - n_2^2\right) = 0 \end{aligned}$$

Equation for normal surfaces

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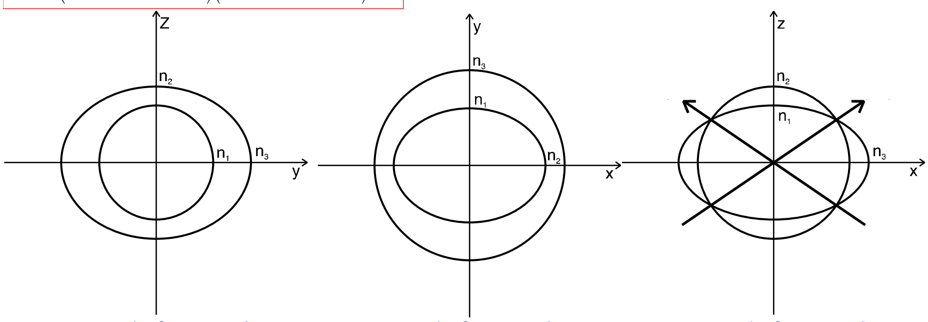
Wave normal surfaces and optic axes

$$\begin{aligned} &n_1^2 x^2 \left(x^2 + y^2 + z^2 - n_2^2\right)\left(x^2 + y^2 + z^2 - n_3^2\right) \\ &+ n_2^2 y^2 \left(x^2 + y^2 + z^2 - n_1^2\right)\left(x^2 + y^2 + z^2 - n_3^2\right) \\ &+ n_3^2 z^2 \left(x^2 + y^2 + z^2 - n_1^2\right)\left(x^2 + y^2 + z^2 - n_2^2\right) = 0 \end{aligned}$$

$$n_3 \geq n_2 \geq n_1$$



Result for $x = 0$ $\quad$ Result for $z = 0$ $\quad$ Result for $y = 0$

$$\begin{cases} y^2 + z^2 = n_1^2 & \text{: circle of radius } n_1 \\ n_2^2 n_3^2 \left(y^2 + z^2\right)\left(\dfrac{y^2}{n_3^2} + \dfrac{z^2}{n_2^2} - 1\right) = 0 & \text{: ellipse with semi-axes } n_2 \text{ and } n_3 \end{cases}$$

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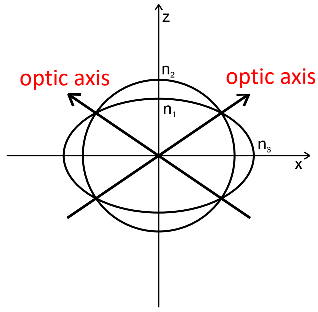
Optic axis

$n_3 \geq n_2 \geq n_1$

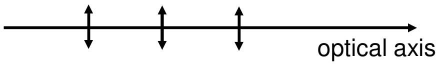
The two curves cross!!!

- Two particular directions where there is only **one value of the phase velocity**
- **Optic axis**



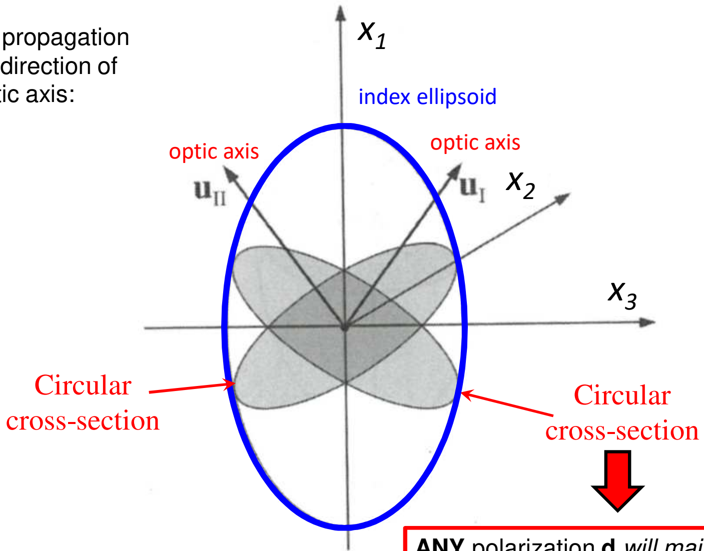
Result for $y = 0$

Note: optic<u>al</u> axis is not the same as the optic axis!!!



Index ellipsoid and optic axes

Wave propagation in the direction of an optic axis:

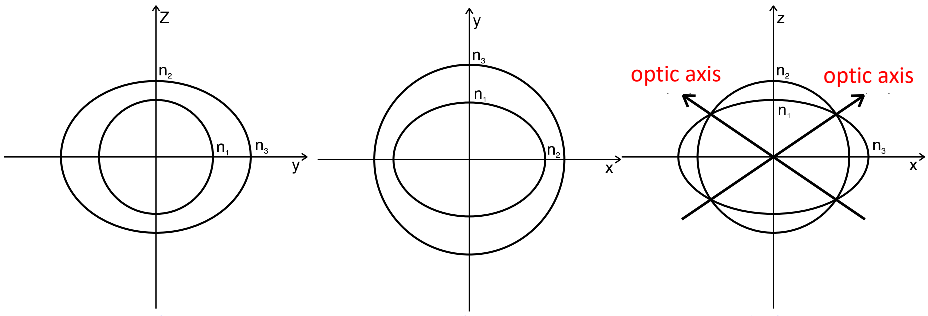


ANY polarization **d** *will maintain its polarization going through the crystal*

Champeau et al., p. 729

Normal surfaces and optic axes

$n_3 \geq n_2 \geq n_1$



Result for $x = 0$ Result for $z = 0$ Result for $y = 0$

General case: $n_3 \neq n_2 \neq n_1$

What if $n_3 = n_2 \neq n_1$?

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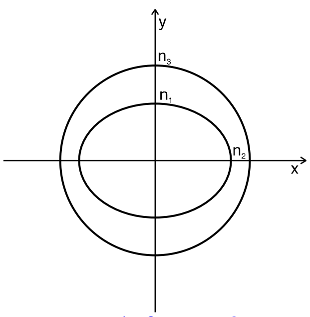
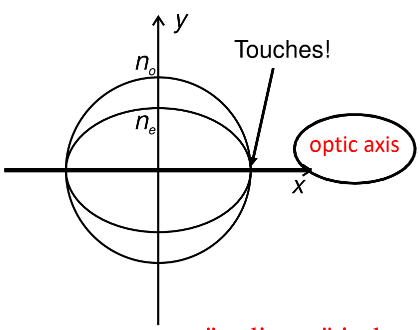
Uniaxial vs biaxial crystals

Recall:



Result for $z = 0$

$n_3 \neq n_2 \neq n_1$



$n_2 = n_3 \equiv n_o$ "ordinary" index

$n_1 \equiv n_e$ "extraordinary" index

$n_o > n_e$

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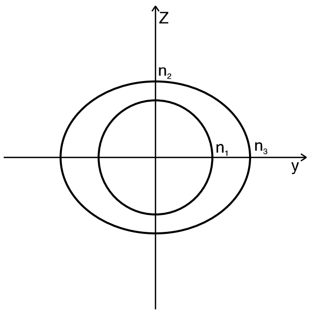
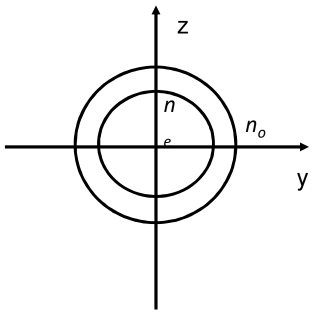
Uniaxial vs biaxial crystals

Recall:



Result for $x = 0$

$$n_3 \neq n_2 \neq n_1$$



$n_2 = n_3 \equiv n_o$ "ordinary" index

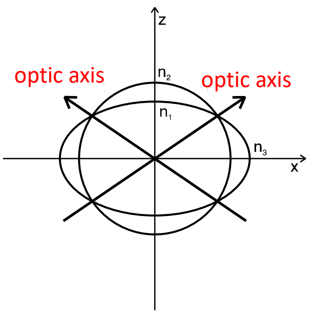
$n_1 \equiv n_e$ "extraordinary" index

$$n_o > n_e$$

Uniaxial vs biaxial crystals

Recall:

biaxial crystal

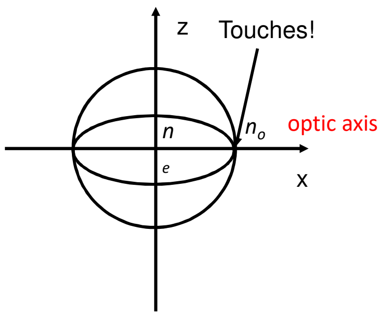


Result for $y = 0$

$$n_3 \neq n_2 \neq n_1$$

2 optic axes

uniaxial crystal



$n_2 = n_3 \equiv n_o$ "ordinary" index

$n_1 \equiv n_e$ "extraordinary" index

$$n_o > n_e$$

$$n_o < n(\mathbf{k}) < n_e$$

1 optic axis

$$\Delta n \equiv n_e - n_o$$

- If $\Delta n < 0$ the material is considered “negative” (e.g., calcite)
- If $\Delta n > 0$ the material is considered “positive” (e.g., quartz)

TABLE 8.1 Refractive Indices of Some Uniaxial Birefringent Crystals (λ_0 = 589.3 nm)

Crystal	n_o	n_e
Tourmaline	1.669	1.638
Calcite	1.6584	1.4864
Quartz	1.5443	1.5534
Sodium nitrate	1.5854	1.3369
Ice	1.309	1.313
Rutile (TiO_2)	2.616	2.903

Hecht, *Optics*

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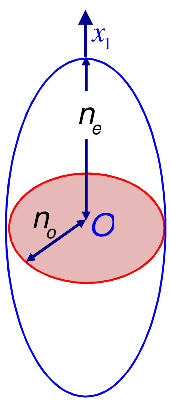
Propagation in a uniaxial crystal

- index ellipsoid: ellipsoid of revolution or spheroid
- 1 circular section only => 1 optic axis
- an incident wave with **u** <u>in the direction of the optic axis</u> will maintain its polarization going through the crystal,

$$n = n_o$$

$$1 = \frac{x_1^2}{n_e^2} + \frac{x_2^2}{n_o^2} + \frac{x_3^2}{n_o^2}$$



$n_2 = n_3 \equiv n_o$ "ordinary" index
$n_1 \equiv n_e$ "extraordinary" index

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Propagation in a uniaxial crystal: arbitrary **u**

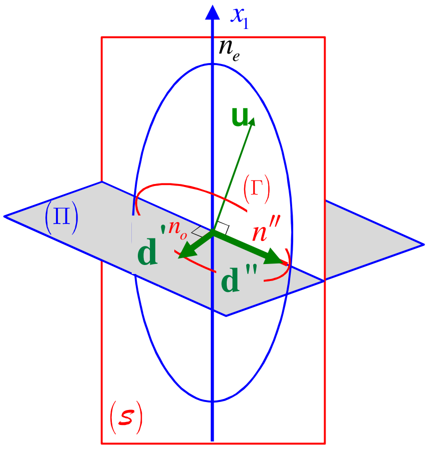
$n_2 = n_3 \equiv n_o$ "ordinary" index
$n_1 \equiv n_e$ "extraordinary" index

x_1: optic axis

$$1 = \frac{x_1^2}{n_e^2} + \frac{x_2^2}{n_o^2} + \frac{x_3^2}{n_o^2}$$

(s) principal plane: formed by **u** and the optic axis

- The index ellipsoid is *symmetrical* about the principal plane
- Plane Π: normal to **u**
- Resulting elliptical cross-section Γ: *symmetrical* about (s)
- Semi-axes of Γ:
  - One *perpendicular* to (s) : n_o, $\mathbf{d}'$
  - One *parallel* to (s) : n'', $\mathbf{d}''$



Propagation in a uniaxial crystal

$n_2 = n_3 \equiv n_o$ "ordinary" index
$n_1 \equiv n_e$ "extraordinary" index

x_1: optic axis

$$1 = \frac{x_1^2}{n_e^2} + \frac{x_2^2}{n_o^2} + \frac{x_3^2}{n_o^2}$$

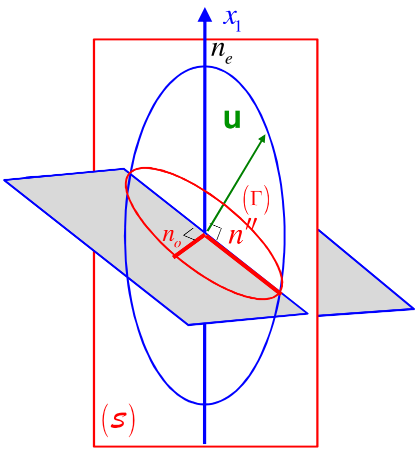
- Semi-axes of Γ:
  - One *perpendicular* to (s) : n_o, $\mathbf{d}'$
  - One *parallel* to (s) : n'', $\mathbf{d}''$

- No matter how **u** is tilted, *perpendicular* semi-axis always has the *same length and direction*

→ **Ordinary wave**

- The length and direction of the *parallel* semi-axis *depends on* **u**

→ **Extraordinary wave**



Propagation in a uniaxial crystal

Ordinary wave

$n_2 = n_3 \equiv n_o$ "ordinary" index
$n_1 \equiv n_e$ "extraordinary" index

x_1: optic axis

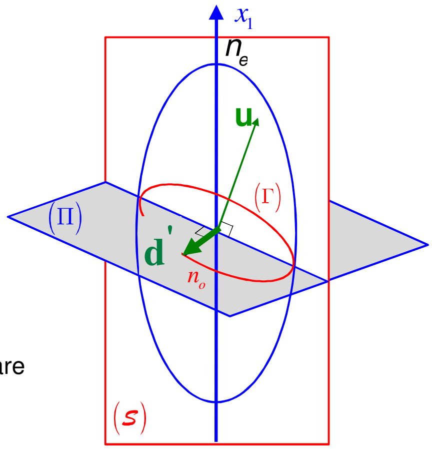
- **d'** is perpendicular to (s)
- $n' = n_o$ no matter the orientation of **u**
- **d'** is in the (x_2, x_3) plane, and $\varepsilon_{r2} = \varepsilon_{r3} = n_o^2$

$$\Rightarrow \quad D_2 = \varepsilon_0 n_o^2 E_2; \quad D_3 = \varepsilon_0 n_o^2 E_3$$

D and E are parallel!

Poynting vector **S** and the wave normal **u** are also parallel

Wave behaves as if it is in an isotropic medium!

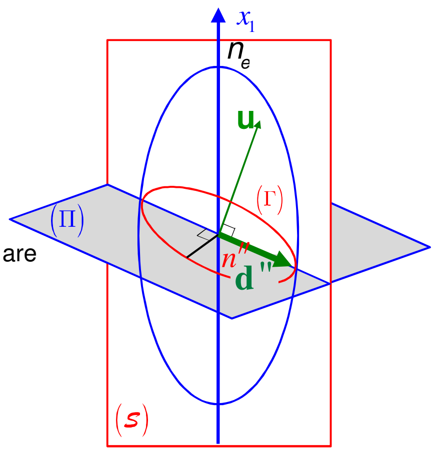


Propagation in a uniaxial crystal

x_1: optic axis

Extraordinary wave

- **d"** is in the plane (s)
- n'' depends on the orientation of **u**
- **D** and **E** are **NOT** parallel
- Poynting vector **S** and the wave normal **u** are **NOT** parallel
- Wave does NOT behave as if it is in an isotropic medium!



Propagation in a uniaxial crystal: determining n''

- C: cross-section of the index ellipsoid in the principal plane
- n'': length of the semi-axis of the ellipse (Γ)

(Γ) is in the plane perpendicular to **u**

⬇

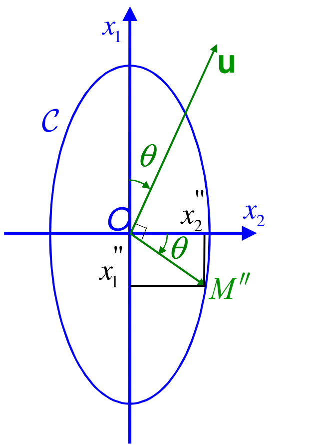
length of the vector **OM''**

coordinates of **OM''**: $x_2'' = n''\cos\theta\ ;\ x_1'' = n''\sin\theta$ (A)

equation of ellipse C: $1 = \frac{x_1^2}{n_e^2} + \frac{x_2^2}{n_o^2}$ (B)

Plug (A) into (B) and solve for n''

$$n'' = 1\Big/\sqrt{\frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2}}$$



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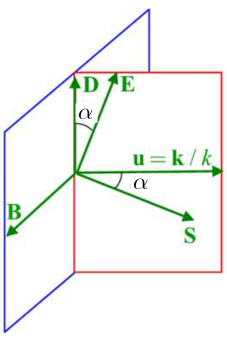
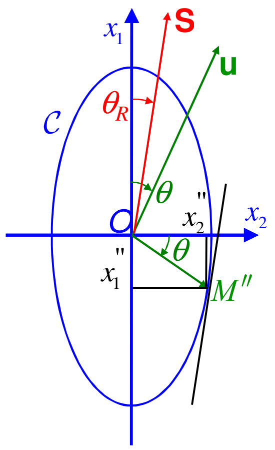
Propagation in a uniaxial crystal: determining the direction of **S**, the Poynting vector

- Recall: **S** in the same plane as **u, E, D**, perpendicular to **B**
- Recall: **S** is tangential to ellipsoid at M''

⬇

Want to find θ_R





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Propagation in a uniaxial crystal: determining the direction of S, the Poynting vector

- Recall: **S** in the same plane as **u, E, D**, perpendicular to **B**
- Recall: **S** is tangential to ellipsoid at M''

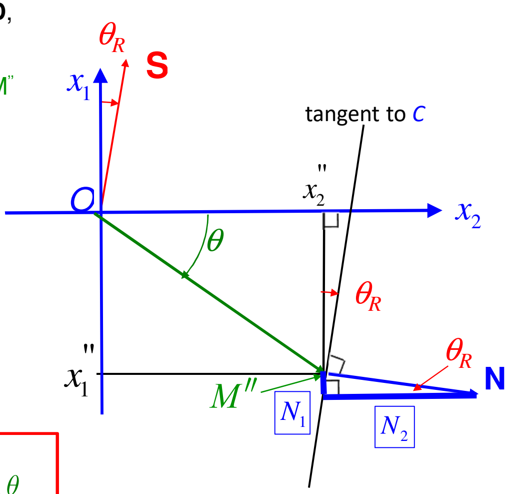
Want to find θ_R

- Find first the normal to *C* at M''

$$g(x_1, x_2) = \frac{x_1^2}{n_e^2} + \frac{x_2^2}{n_o^2} - 1$$

$$\mathbf{N} = \begin{pmatrix} \partial g / \partial x_1 \\ \partial g / \partial x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 / n_e^2 \\ 2x_2 / n_o^2 \end{pmatrix} \equiv \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

$$\tan\theta_R = \frac{N_1}{N_2} = \frac{x_1'' / n_e^2}{x_2'' / n_o^2} = \frac{n_o^2}{n_e^2}\tan\theta$$



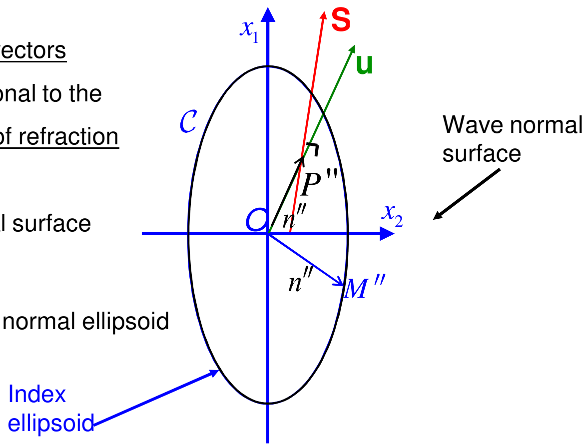
Index ellipsoid and wave normal surfaces

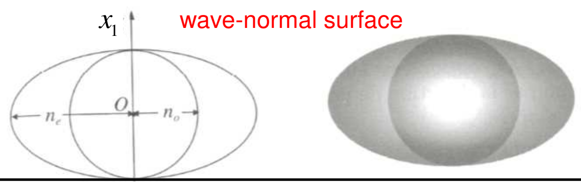
Recall: wave-normal surface:

- For each **u,** associate two vectors whose lengths are proportional to the two corresponding indices of refraction *n'* and *n''*

P'' is on the normal surface

S is perpendicular to wave normal ellipsoid





Snell's (Descartes'?) laws for anisotropic media

According to Dijksterhuis,[12] "In *De natura lucis et proprietate* (1662) Isaac Vossius said that *Descartes had seen Snell's paper and concocted his own proof*. We now know this charge to be undeserved but it has been adopted many times since." *Both Fermat and Huygens repeated this accusation that Descartes had copied Snell.* In French, Snell's Law is called "la loi de Descartes" or "loi de Snell-Descartes."

Wikipedia *(italics and* underlining mine)

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Snell's laws

- Basic premise: the tangential component of the wave vector **k** must be continuous across an interface

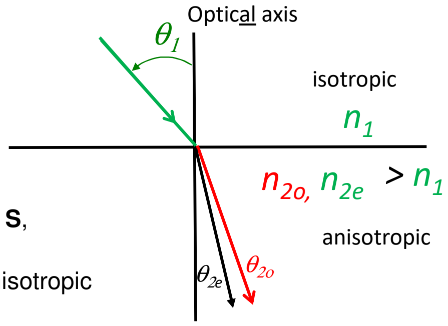
Recall Snell's laws for isotropic media:

- The wave normals (**u**) of the *incident*, *refracted* waves and the normal to the interface are all in the same plane (the plane of incidence)

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Second medium anisotropic (uniaxial):

$$n_1 \sin\theta_1 = n_{2o} \sin\theta_{2o} = n_{2e} \sin\theta_{2e}$$



Note:

- Snell's laws apply to **k** and NOT to **S**, the Poynting vector
- n'' changes with direction in the anisotropic medium!

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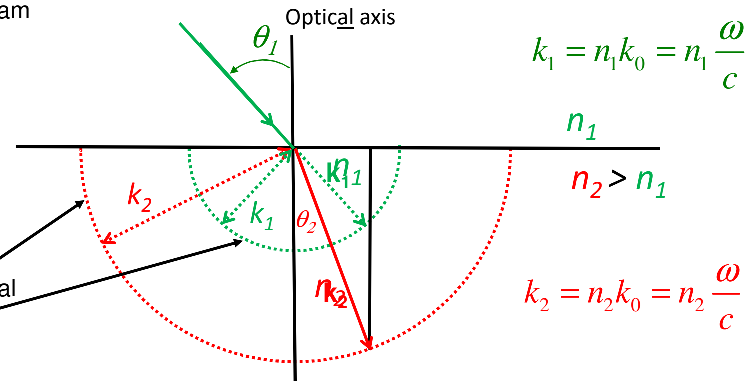
Graphical method for applying Snell's laws: two *isotropic* media

Method:

- Draw half circle with radius k_1 ➡ <u>the tangential components are equal</u>
- Draw half circle with radius k_2

Rescale diagram



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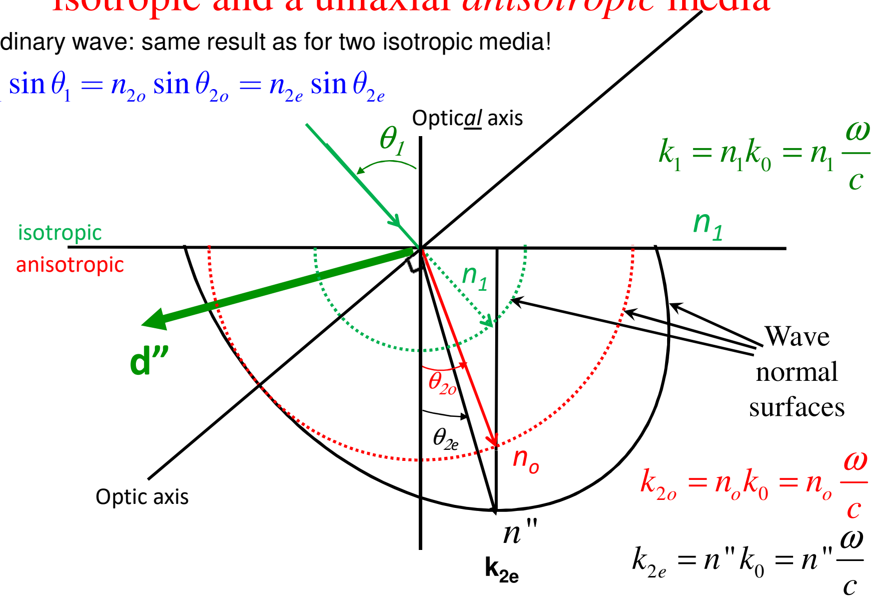
Graphical method for Snell's laws: an isotropic and a uniaxial *anisotropic* media

Ordinary wave: same result as for two isotropic media!

$n_1 \sin\theta_1 = n_{2o} \sin\theta_{2o} = n_{2e} \sin\theta_{2e}$



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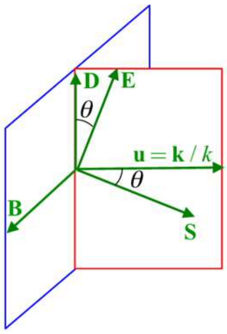
Main points!

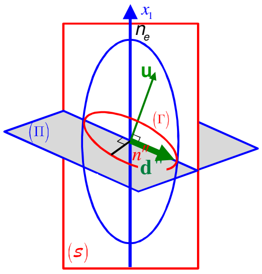


In an anisotropic medium:

$$\mathbf{D} = \varepsilon_0 \overleftrightarrow{\boldsymbol{\varepsilon}}_{\mathbf{r}} \mathbf{E}$$

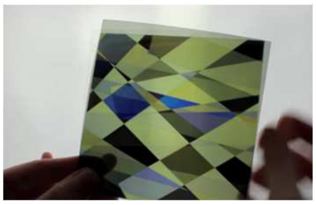
Tensor!

- Wave vector **k** is (in general) **NOT** in the same direction as the **propagation of energy**



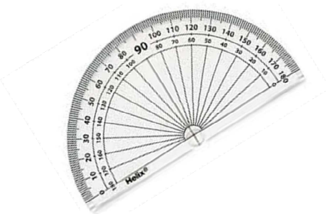
- <u>**Two waves**</u>! unless…
- Graphical method for finding indices and polarizations
- Wave normal surfaces for refraction

Lecture 7: Experiments



- Experiments:
  - Place Scotch tape between crossed polarizers
  - Rotate Scotch tape

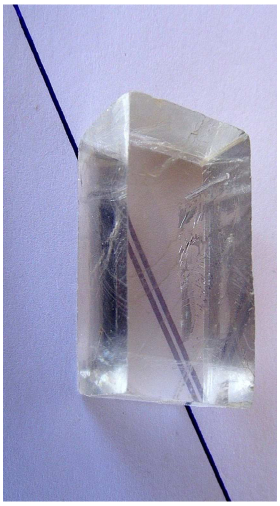
  ⇒ <u>observations</u>



- Place a clear plastic object (e.g. protractor) between crossed polarizers

=> <u>observations</u>

Experiment!



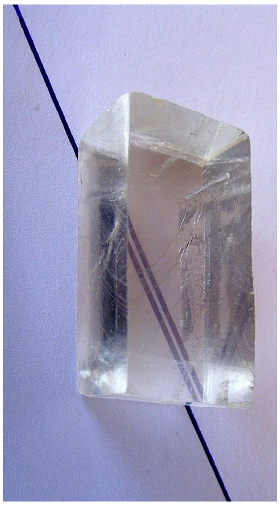
- Experiment:
  - Draw a black dot on a piece of paper
  - Place calcite crystal on top
  - Rotate crystal

There are two dots! One is stationary, the other rotates as the crystal rotates.

Stationary dot: ordinary wave
Rotating dot: extraordinary wave

Extraordinary wave Poynting vector changes direction as the optic axis direction is changed!

Experiment

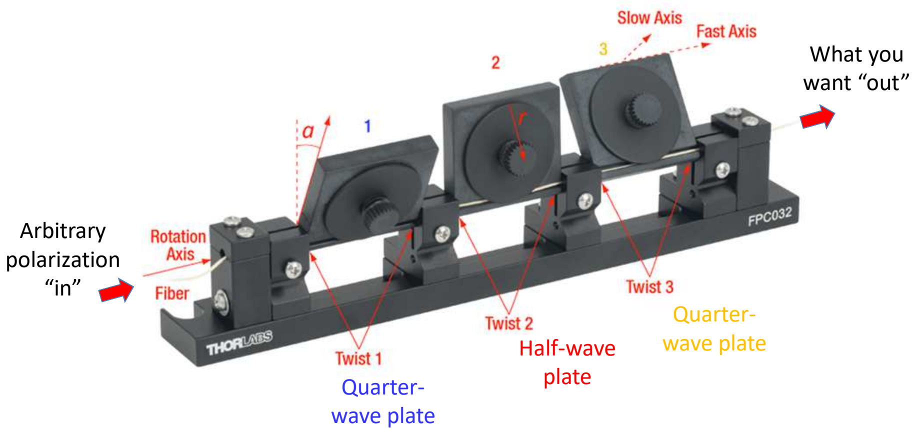


- Experiment:
  - Look through a polarizer at crystal
  - Rotate polarizer

One or the other of the black dots is visible.

Polarization directions of the ordinary and extraordinary waves are perpendicular!!!

Controlling the polarization of light



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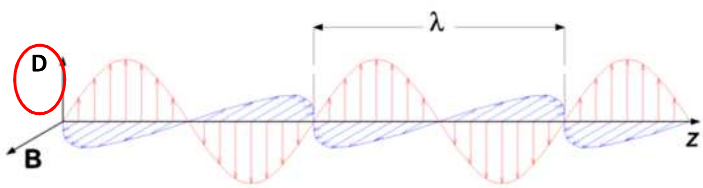
Outline

- Polarization states of light—mathematical description
- Jones vectors
- Manipulation and control of the polarization of light



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Polarization

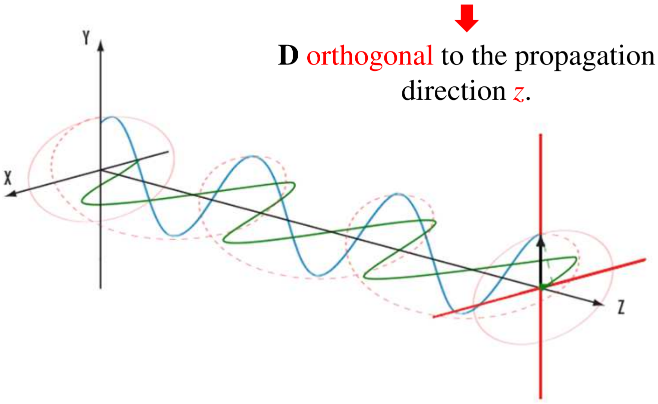
Polarization:

- direction and variation of the electric displacement vector **D** during propagation

- **Monochromatic plane wave** in a transparent medium

⬇

D orthogonal to the propagation direction z.



https://www.edmundoptics.com/resources/application-notes/optics/introduction-to-polarization/

Polarization

- **Monochromatic sinusoidal plane wave** in a transparent medium

⬇

D orthogonal to propagation direction z.

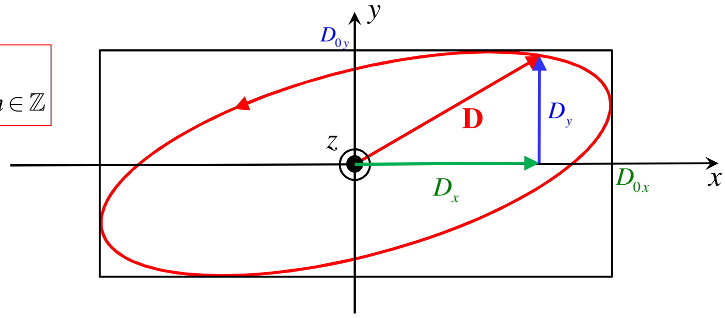
$$D_x = D_{0x}\cos(\omega_0 t - kz - \psi_x)$$
$$D_y = D_{0y}\cos(\omega_0 t - kz - \psi_y)$$

$D_{0x}, D_{0y} \geq 0$ Amplitudes

ψ_x, ψ_y Phases, constant

$D_{0x} \neq D_{0y}$

$\psi_y - \psi_x \neq m\pi,\ m \in \mathbb{Z}$



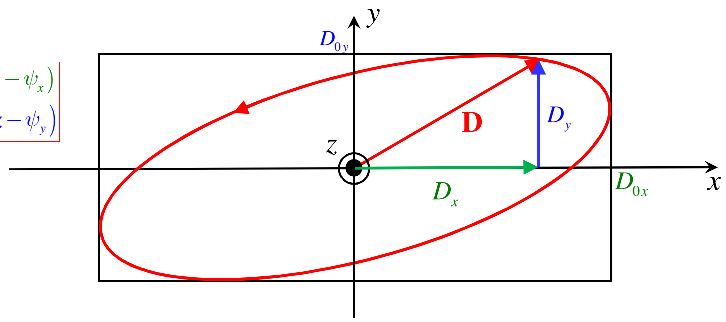
Elliptical polarisation

State of polarization

The *state of polarization* is determined by

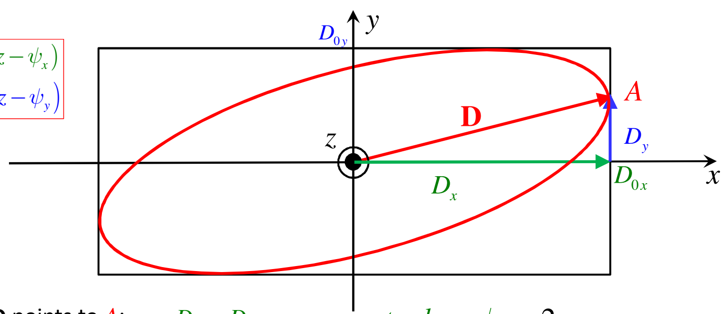
- $\frac{D_{0y}}{D_{0x}}$ OR • $\frac{D_{0y}}{D_{0x}}$ • $\psi_y - \psi_x$
- angle of ellipse axis
- rotation *direction*

$$D_x = D_{0x}\cos\left(\omega_0 t - kz - \psi_x\right)$$
$$D_y = D_{0y}\cos\left(\omega_0 t - kz - \psi_y\right)$$



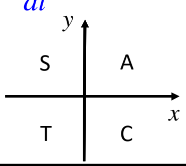
Determining the rotation direction of the polarization ellipse

$$D_x = D_{0x}\cos\left(\omega_0 t - kz - \psi_x\right)$$
$$D_y = D_{0y}\cos\left(\omega_0 t - kz - \psi_y\right)$$



- Consider when **D** points to *A*: $D_x = D_{0x}$ $\omega_0 t - kz - \psi_x = 2m\pi$
- In order to know the rotation direction, we need to know the sign of $\frac{dD_y}{dt}$

$$\frac{dD_y}{dt} = -\omega_0 D_{0y}\sin\left(\omega_0 t - kz - \psi_y\right) \quad = \omega_0 D_{0y}\sin\left(\psi_y - \psi_x\right)$$

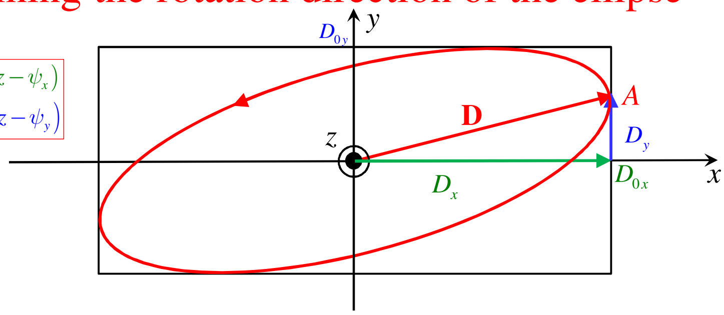


$0 < \psi_y - \psi_x < \pi \Leftrightarrow$ counter-clockwise

$-\pi < \psi_y - \psi_x < 0 \Leftrightarrow$ clockwise

Determining the rotation direction of the ellipse

$D_x = D_{0x}\cos(\omega_0 t - kz - \psi_x)$
$D_y = D_{0y}\cos(\omega_0 t - kz - \psi_y)$



$$\frac{dD_y}{dt} = -\omega_0 D_{0y}\sin(\omega_0 t - kz - \psi_y) \quad = \omega_0 D_{0y}\sin(\psi_y - \psi_x)$$

$0 < \psi_y - \psi_x < \pi \Leftrightarrow$ left elliptically polarized
$-\pi < \psi_y - \psi_x < 0 \Leftrightarrow$ right elliptically polarized

counter-clockwise rotation as a function of time when the wave is travelling ***towards*** the observer

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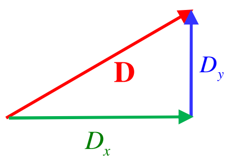
Polarization: special cases

$D_x = D_{0x}\cos(\omega_0 t - kz - \psi_x)$
$D_y = D_{0y}\cos(\omega_0 t - kz - \psi_y)$

$D_{0x}, D_{0y} \geq 0$

If $\psi_y - \psi_x = 0$ or π

➡ components are _in phase_ or _out of phase_



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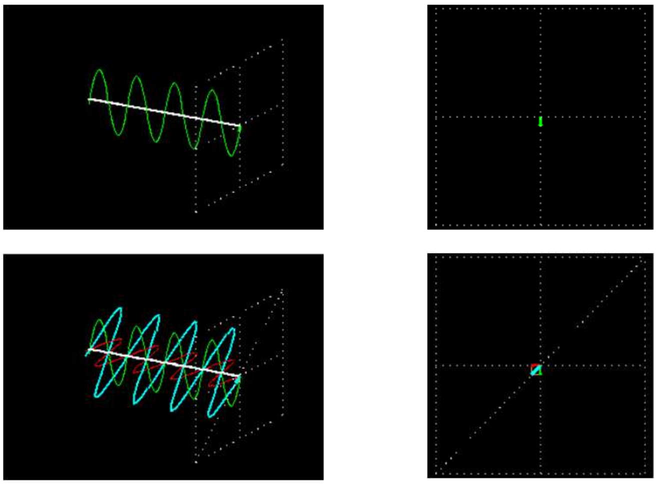
Linear polarization



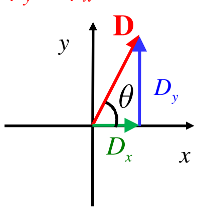
http://cddemo.szialab.org/

Linear polarization

$$D_x = D_{0x}\cos\left(\omega_0 t - kz - \psi_x\right)$$
$$D_y = D_{0y}\cos\left(\omega_0 t - kz - \psi_y\right)$$

$$D_{0x}, D_{0y} \geq 0$$

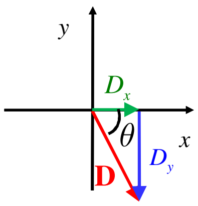
If $\psi_y - \psi_x = 0$:



$$\text{If } \psi_y - \psi_x = 0: \tan\theta = \frac{D_y}{D_x} = \frac{D_{0y}}{D_{0x}}.$$

$$\text{If } \psi_y - \psi_x = \pi: \tan\theta = \frac{D_y}{D_x} = -\frac{D_{0y}}{D_{0x}}.$$

If $\psi_y - \psi_x = \pi$:



D makes an angle $\pm\theta$ with the x axis.

Circular polarization

What if $\psi_y - \psi_x = \pm\,\pi/2$ and $D_{0x} = D_{0y}$?

$$D_x = D_{0x}\cos(\omega_0 t - kz - \psi_x)$$
$$D_y = D_{0y}\cos(\omega_0 t - kz - \psi_y)$$

$\Rightarrow$

$$D_x = +D_{0x}\cos(\omega_0 t - kz - \psi_x)$$
$$D_y = \pm D_{0x}\sin(\omega_0 t - kz - \psi_x)$$

The end of **D** draws a circle of radius D_{0x}.

- If $\psi_y - \psi_x = +\,\pi/2$: left circular polarization
- If $\psi_y - \psi_x = -\,\pi/2$: right circular polarization

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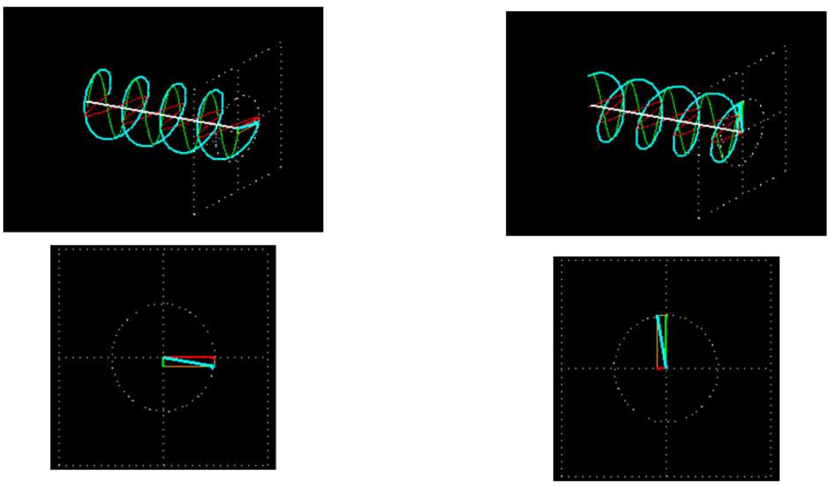
Circular polarization



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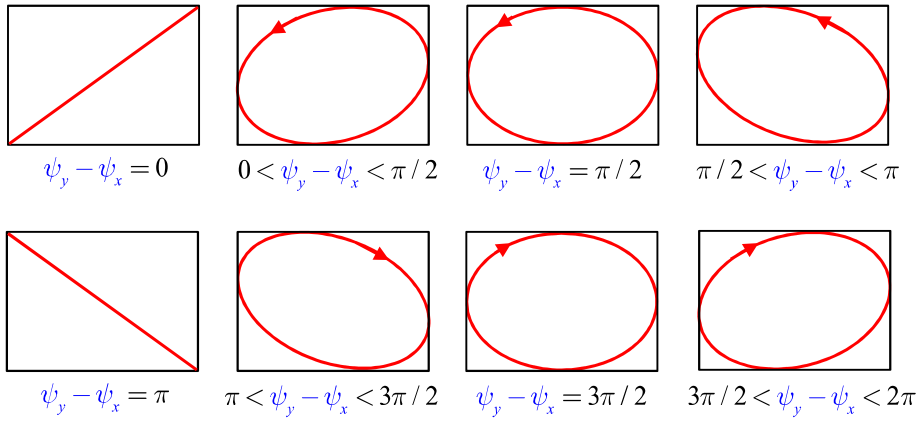
Diverse polarization states



$$D_x = D_{0x}\cos\left(\omega_0 t - kz - \psi_x\right)$$
$$D_y = D_{0y}\cos\left(\omega_0 t - kz - \psi_y\right)$$

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Jones vectors

Handy mathematical formulism for describing and manipulating polarization states

Write
$$D_x = D_{0x}\cos\left(\omega_0 t - kz - \psi_x\right)$$
$$D_y = D_{0y}\cos\left(\omega_0 t - kz - \psi_y\right)$$
in complex notation, i.e.,

$$\mathcal{D}_x = D_{0x}\exp\left[i(kz - \omega t + \psi_x)\right]$$
$$\mathcal{D}_y = D_{0y}\exp\left[i(kz - \omega t + \psi_y)\right]$$

$$\mathcal{D}_{0x} \equiv D_{0x}\exp\left[i\psi_x\right]$$
$$\mathcal{D}_{0y} \equiv D_{0y}\exp\left[i\psi_y\right]$$

Define the Jones vector for the polarization state as:

$$\mathbf{u} = \begin{pmatrix} \mathcal{D}_{0x} \\ \mathcal{D}_{0y} \end{pmatrix} = \begin{pmatrix} D_{0x}\exp\left[i\psi_x\right] \\ D_{0y}\exp\left[i\psi_y\right] \end{pmatrix}$$

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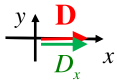
Jones vectors

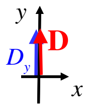
$$\mathbf{u} = \begin{pmatrix} \mathcal{D}_{0x} \\ \mathcal{D}_{0y} \end{pmatrix} = \begin{pmatrix} D_{0x} \exp[i\psi_x] \\ D_{0y} \exp[i\psi_y] \end{pmatrix}$$

Jones vectors for linear polarization:

Linear polarization oriented along the Ox axis: $\mathbf{u}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

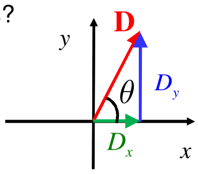


Linear polarization oriented along the Oy axis: $\mathbf{u}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



Orthonormal basis!

What is the Jones vector for linear polarization at an angle θ with respect to the Ox axis?



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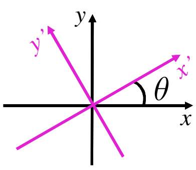
Jones vectors

$$\mathbf{u} = \begin{pmatrix} \mathcal{D}_{0x} \\ \mathcal{D}_{0y} \end{pmatrix} = \begin{pmatrix} D_{0x} \exp[i\psi_x] \\ D_{0y} \exp[i\psi_y] \end{pmatrix}$$



Recall from math class:

$$\hat{x}' = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$
$$\hat{y}' = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Jones vector <u>for linear polarization at an angle θ with respect to the Ox axis</u>:

$$\mathbf{u}_\theta = R(\theta)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

(normalized) Jones vectors for <u>**circular polarization**</u>?

Recall: $\psi_y - \psi_x = \pm\pi/2$ and $D_{0x} = D_{0y}$

$$\mathbf{u} = \begin{pmatrix} \mathcal{D}_{0x} \\ \mathcal{D}_{0y} \end{pmatrix} = \begin{pmatrix} D_{0x} \exp[i\psi_x] \\ D_{0x} \exp\left[i\psi_x \pm \frac{\pi}{2}\right] \end{pmatrix} = D_{0x} \exp[i\psi_x] \begin{pmatrix} 1 \\ e^{\pm i\left(\frac{\pi}{2}\right)} \end{pmatrix} \propto \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

<u>**circular polarization**</u>

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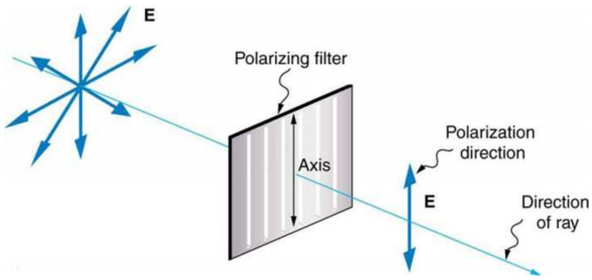
Jones vectors for circularly polarized light

Left circular polarization: $\mathbf{u}_L = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ +i \end{pmatrix}$

Right circular polarization: $\mathbf{u}_R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

Orthonormal basis!

Polarizers and the Jones formulism



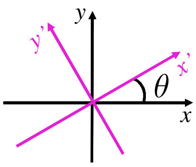
Ideal polarizer whose transmission axis is aligned with Ox:

$$\mathbf{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

What about the Jones matrix for an ideal polarizer whose transmission axis is at an angle θ to the Ox axis?

Polarizers in the Jones formulism

What about the Jones matrix for an ideal polarizer whose transmission axis is at an angle θ to the Ox axis?



$$\hat{x} = \cos\theta\,\hat{x}' - \sin\theta\,\hat{y}'$$
$$\hat{y} = \sin\theta\,\hat{x}' + \cos\theta\,\hat{y}'$$

Recall: $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ gives the coordinates of the new axes $Ox'y'$ in terms of the *old* coordinates Oxy

1. Find *old* axes in terms of the *new* coordinate system

$$R^{-1}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Thus, the incoming polarization $\begin{pmatrix} D_x \\ D_y \end{pmatrix}$ in terms of the coordinate system of the polarizer is

$$R^{-1}(\theta)\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} D_x \\ D_y \end{pmatrix}$$

Polarizers in the Jones formulism

2. Next, apply the effect of the polarizer:

$$\mathbf{P}_0 R^{-1}(\theta)\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} D_x \\ D_y \end{pmatrix}$$

3. Finally, express in terms of the original coordinate system

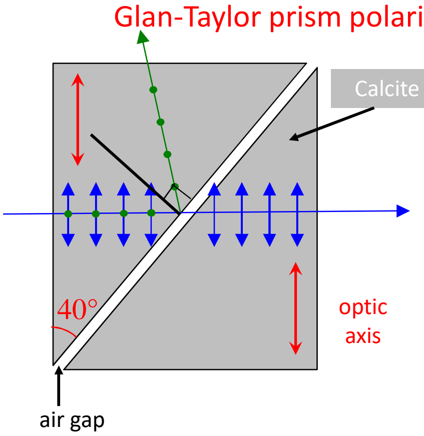
$$\begin{aligned} \mathbf{P}_\theta &= R(\theta)\mathbf{P}_0 R^{-1}(\theta) \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \end{aligned}$$

Jones matrix for an ideal polarizer whose transmission axis is at an angle θ to the Ox axis

Controling the polarizaton

Goal: light with arbitrary polarization "in", linearly polarized light "out".

Glan-Taylor prism polarizer



Ordinary wave polarization:

➡ out of screen $n_o = 1.658$

Extraordinary wave polarization:

➡ parallel to optic axis $n_e = 1.486$

$$\theta_{crit\text{-}o} = \arcsin(1/n_o) = 37°$$

$$\theta_{crit\text{-}e} = \arcsin(1/n_e) = 42°$$

Extraordinary wave continues!
Ordinary wave is reflected!

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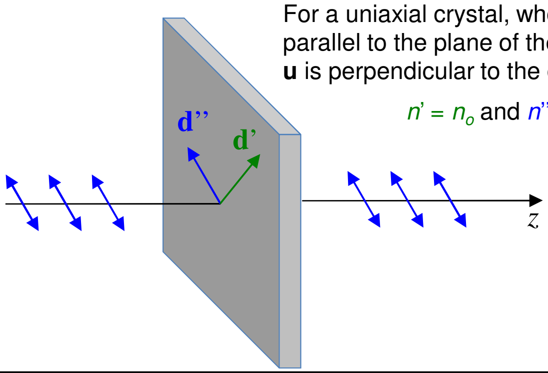
Controling the polarization of light

- Use polarizers
- Use a birefringent optical flat ➡ "**wave plates**" or "**retarders**"
  - Normal incidence
  - Walk-off is negligible

Recall:
- **d'** and **d"**, special polarization directions for which the polarization is maintained as the light propagates
- associated indices n' and n''

For a uniaxial crystal, whose optic axis is parallel to the plane of the optical flat (i.e., **u** is perpendicular to the optic axis)

$n' = n_o$ and $n'' = n_e$



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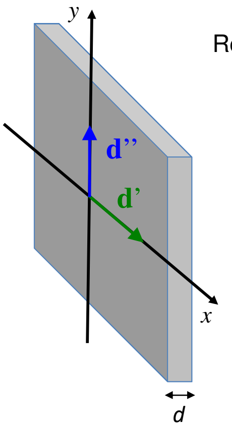
What happens to the polarization as the light propagates through a birefringent crystal?



Choose x and y to be along **d'** and **d''** respectively

Recall: $D_x = D_{0x}\cos(\omega_0 t - kz - \psi_x)$ $\quad D_y = D_{0y}\cos(\omega_0 t - kz - \psi_y)$

$$\mathbf{u} = \begin{pmatrix} \mathcal{D}_{0x} \\ \mathcal{D}_{0y} \end{pmatrix} = \begin{pmatrix} D_{0x}\exp[i\psi_x] \\ D_{0y}\exp[i\psi_y] \end{pmatrix}$$

Each component travels at a different speed!

$$\mathcal{D}_x = \mathcal{D}_{0x}\exp(ik'd) = \mathcal{D}_{0x}\exp\left(i\frac{2\pi}{\lambda}n'd\right)$$

$$\mathcal{D}_y = \mathcal{D}_{0y}\exp(ik''d) = \mathcal{D}_{0y}\exp\left(i\frac{2\pi}{\lambda}n''d\right)$$

Note that **if** $n' < n''$, **d'** and **d''** are called the **fast** and **slow** axes respectively

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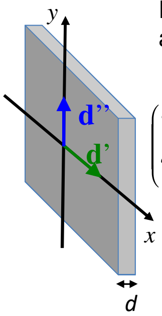
What happens to the polarization as the light propagates through a birefringent crystal?



Find Jones matrix corresponding to propagation through a birefringent crystal of thickness *d* $\quad \mathbf{u} = \begin{pmatrix} \mathcal{D}_{0x} \\ \mathcal{D}_{0y} \end{pmatrix}$

$$\begin{pmatrix} \mathcal{D}_x \\ \mathcal{D}_y \end{pmatrix} = \begin{pmatrix} \mathcal{D}_{0x}\exp\left(i\frac{2\pi}{\lambda}n'd\right) \\ \mathcal{D}_{0y}\exp\left(i\frac{2\pi}{\lambda}n''d\right) \end{pmatrix} = \exp\left(i\frac{2\pi}{\lambda}n'd\right)\begin{pmatrix} \mathcal{D}_{0x} \\ \mathcal{D}_{0y}\exp\left(i\frac{2\pi}{\lambda}(n''-n')d\right) \end{pmatrix}$$

Let $\varphi = \frac{2\pi}{\lambda}(n''-n')d$ $\quad$ Phase delay

$$\begin{pmatrix} \mathcal{D}_x \\ \mathcal{D}_y \end{pmatrix} = \exp\left(i\frac{2\pi}{\lambda}n'd\right)\begin{pmatrix} \mathcal{D}_{0x} \\ \mathcal{D}_{0y}e^{i\varphi} \end{pmatrix}$$

Jones matrix for a wave plate:

$$\mathbf{M} \propto \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

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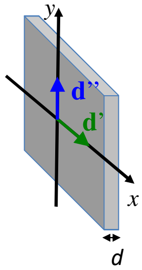
$\mathbf{M} \propto \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$

Quarter wave plate



A phase of 2π corresponds to so a quarter wave plate corresponds to a phase delay of

Choose thickness *d* such that

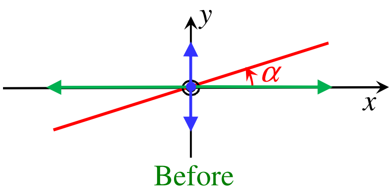
Phase delay: $\varphi = \frac{2\pi}{\lambda}(n_e - n_o)d = \frac{\pi}{2}$

Example: quartz at 560 nm: $(n_e - n_o) = 0.0091$ $\quad d = \frac{\lambda}{4(n_e - n_o)} = 15.4\,\mu\text{m}$

$$\mathbf{M} \propto \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \propto \begin{pmatrix} \exp(-i\pi/4) & 0 \\ 0 & \exp(i\pi/4) \end{pmatrix}$$

$\varphi = \frac{\pi}{2}$

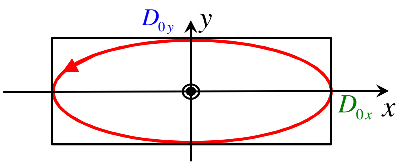
How does a quarter wave plate change the polarization of a linearly polarized wave?



Before

$\varphi = \frac{2\pi}{\lambda}(n_e - n_o)d = \frac{\pi}{2}$

$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix}$



After

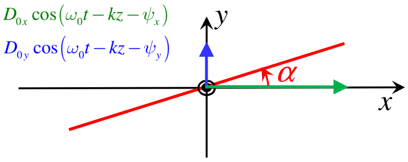
➡ **left** elliptically polarized light!!!

$\psi_y - \psi_x = \frac{\pi}{2}; \quad \tan\alpha = \frac{D_{0y}}{D_{0x}}$

How does a quarter wave plate change the polarization of a linearly polarized wave?

$D_x = D_{0x}\cos(\omega_0 t - kz - \psi_x)$

$D_y = D_{0y}\cos(\omega_0 t - kz - \psi_y)$



$$\varphi = \frac{2\pi}{\lambda}(n_e - n_o)d = \frac{\pi}{2} \qquad \mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix}$$

What is the Jones vector for the initial state?

Before

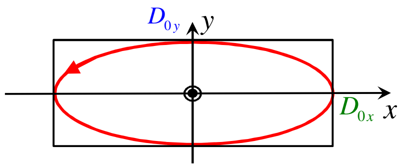
$$\begin{pmatrix} \mathcal{D}_x \\ \mathcal{D}_y \end{pmatrix} = R(\alpha)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

Apply the quarter wave plate:

After

$$\begin{pmatrix} \mathcal{D}_x \\ \mathcal{D}_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ i\sin\alpha \end{pmatrix}$$

$$\begin{cases} \psi_x = 0 \\ \psi_y = \pi/2 \end{cases}$$



$$D_x = \cos\alpha\cos(\omega_0 t - kz - 0) = \cos\alpha\cos(\omega_0 t - kz)$$

$$D_y = \sin\alpha\cos\left(\omega_0 t - kz - \frac{\pi}{2}\right) = \sin\alpha\sin(\omega_0 t - kz)$$

$$\psi_y - \psi_x = \frac{\pi}{2}; \quad \tan\alpha = \frac{D_{0y}}{D_{0x}}$$

elliptically polarized light

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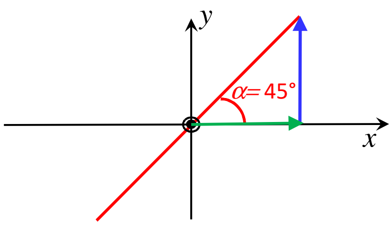
Quarter wave plate: special case, α = π/4



$$\cos\alpha = \sin\alpha = \frac{1}{\sqrt{2}}$$

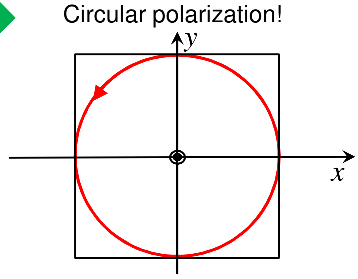
amplitudes are equal!

Recall: quarter wave plate adds π/2 phase delay.

Equal amplitudes + π/2 phase delay gives → Circular polarization!

$$D_x = \cos\alpha\cos(\omega_0 t - kz - 0) = \frac{1}{\sqrt{2}}\cos(\omega_0 t - kz)$$

$$D_y = \sin\alpha\cos\left(\omega_0 t - kz - \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\sin(\omega_0 t - kz)$$



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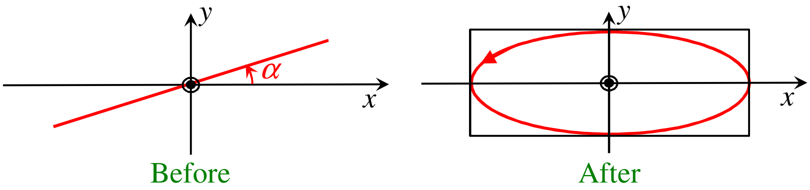
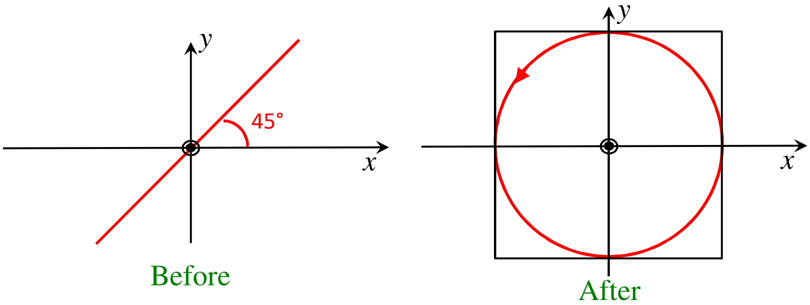
Quarter wave plate

Input wave linearly polarized:



Input wave linearly polarized, α=π/4:



Half wave plate

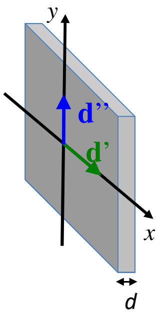
A phase of 2π corresponds to so a half wave plate corresponds to a phase delay of

Choose thickness *d* such that

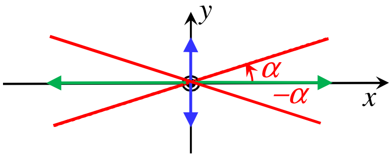
Phase delay: $$\varphi = \frac{2\pi}{\lambda}\left(n_e - n_o\right)d = \pi$$

$$\mathbf{M} \propto \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \propto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$\varphi = \pi$

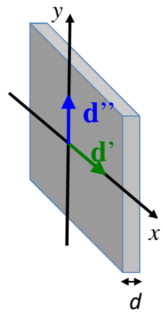


How does a half wave plate change the polarization of a linearly polarized wave?



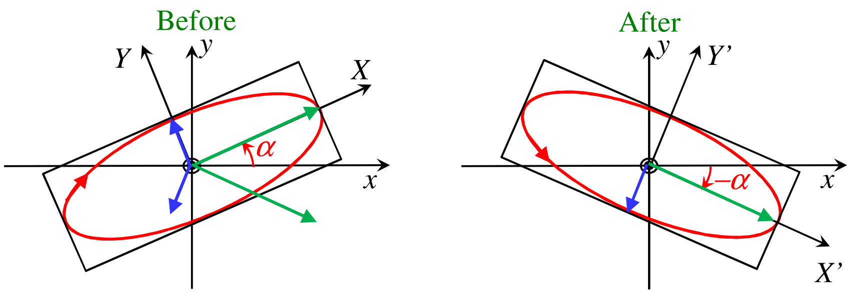
$$\varphi = \frac{2\pi}{\lambda}(n_e - n_o)d = \pi$$

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix}$$



Get **linearly** polarized light "rotated" by 2α symmetrical with respect to $\mathbf{d}'$, $\mathbf{d}''$

How does a half wave plate change the polarization of an elliptically polarized wave?

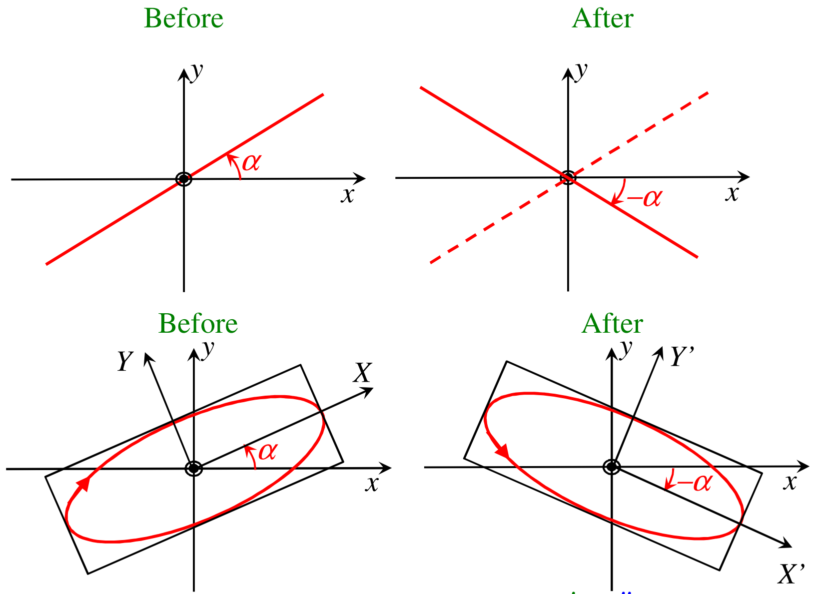


- Consider elliptical polarization as a sum of two linear polarizations
- Submit each y component of each linear polarization to a π delay

- Right polarization becomes left and vice versa!

Half wave plate

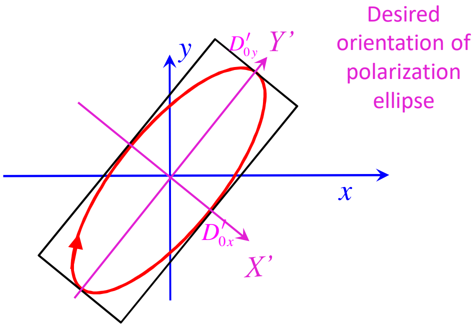
Wave plate axes
= x and y
= $\mathbf{d}'$, $\mathbf{d}''$



Symmetry with respect to $\mathbf{d}'$, $\mathbf{d}''$

Producing the desired state of polarization

Goal: elliptical polarization with a specific orientation and axis ratio

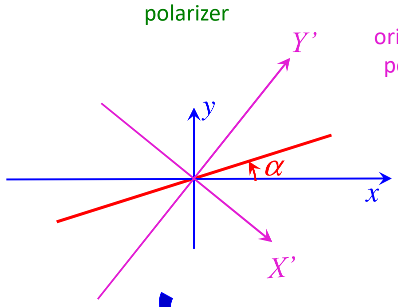


$$\frac{D'_{0y}}{D'_{0x}} = r$$

Producing the desired state of polarization

1. Produce linearly polarized light with a polarizer

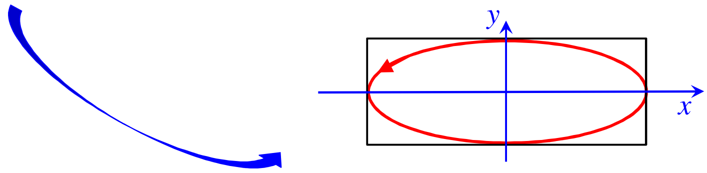
Desired orientation of polarization ellipse



2. Next step: use a quarter wave plate to change polarization to elliptical with the desired "aspect ratio" r

Orient quarter wave plate axes to get

$$r = \tan\alpha$$



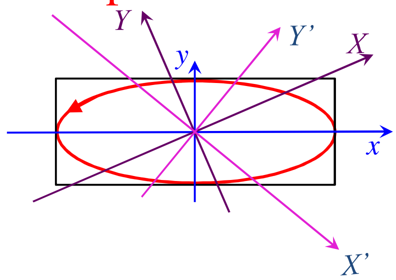
Elliptical polarization with the right "shape" but wrong orientation

Producing the desired state of polarization

3. Next: use a half wave plate to change the polarization ellipse orientation

What should the half wave plate orientation be?

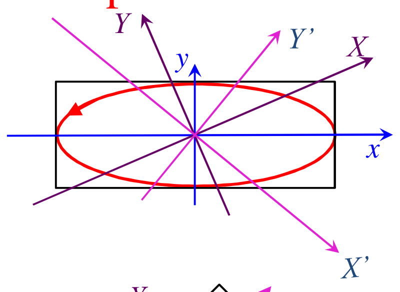
Half wave plate axis should bisect the angle between the current and desired ellipse axes!



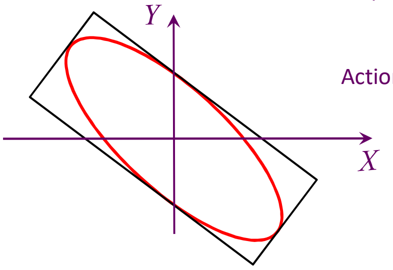
Producing the desired state of polarization

3. Next: use a half wave plate to change the polarization ellipse orientation

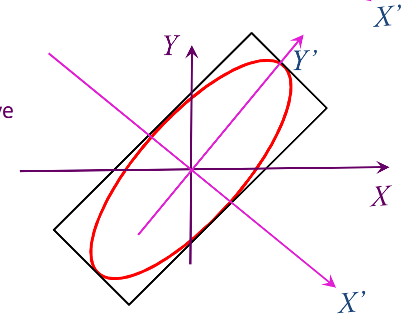
What should the half wave plate orientation be?



Half wave plate axis should bisect the angle between the current and desired ellipse axes!



Action of half wave plate



Polarization control in optical fibres

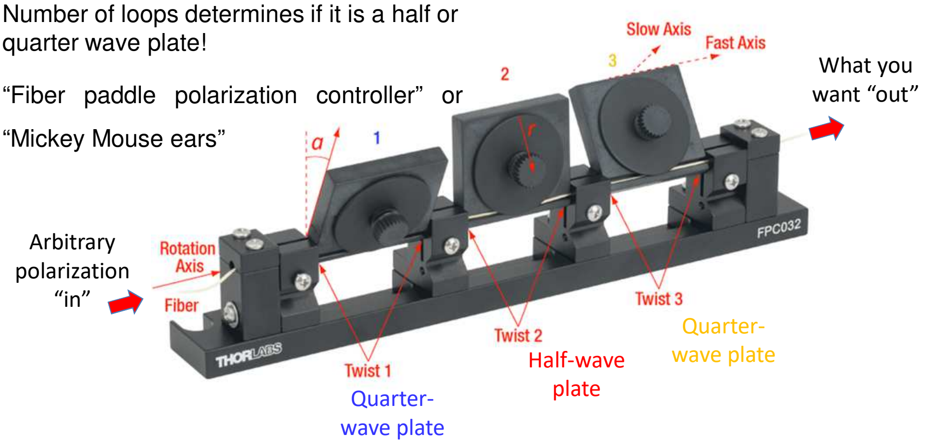
Optical fibres are made of isotropic media (glass). However, optical fibres exhibit *stress-induced birefringence*

When they are coiled (looped), they become anisotropic!

Can easily make quarter and half wave plates by looping fibre!

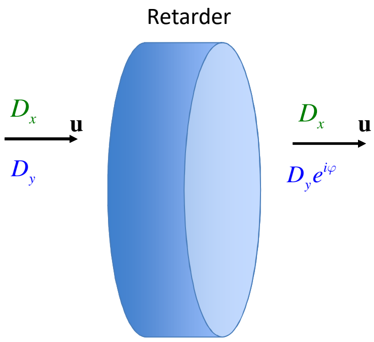
Number of loops determines if it is a half or quarter wave plate!

"Fiber paddle polarization controller" or "Mickey Mouse ears"



https://youtu.be/5O7TL2SUAlo

Interference and retarders



- D_y and D_x are **coherent** (from the same light!) and have now a definite **phase difference**. Sounds like the right conditions for..

- **However**, D_y and D_x are perpendicular to each other so they cannot interfere…

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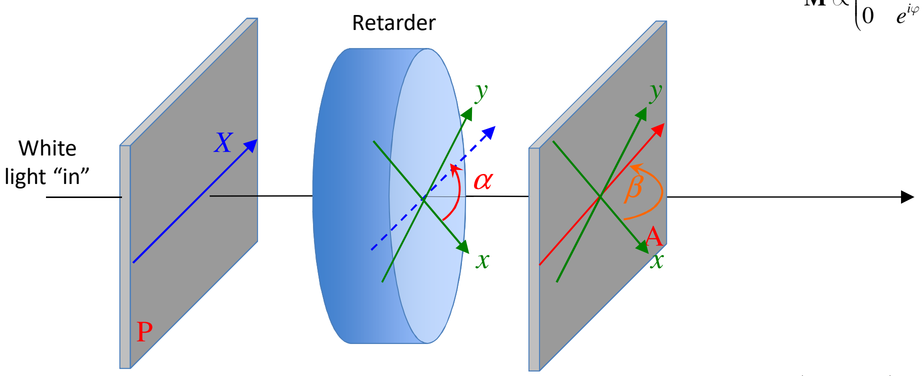
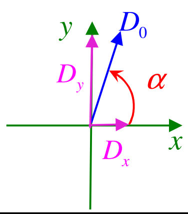
Interference and retarders

$$\mathbf{M} \propto \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$



- Components of incident wave projected on the retarder axes $\begin{pmatrix} D_0 \cos\alpha \\ D_0 \sin\alpha \end{pmatrix}$



After retarder: $\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \begin{pmatrix} D_0 \cos\alpha \\ D_0 \sin\alpha \exp(i\varphi) \end{pmatrix}$

- Place another polarizer (the "analyzer" A), at an angle β to the x axis of the retarder

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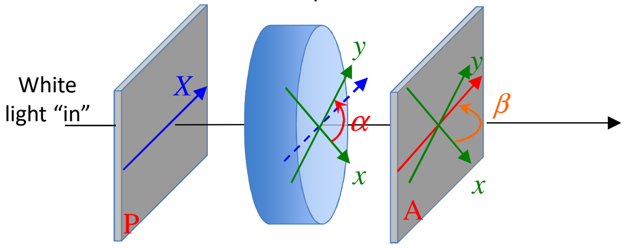
Interference and retarders

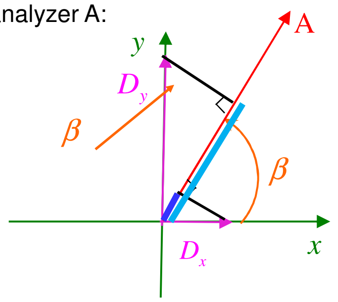
- Two methods for next step:



After retarder:

$$\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \begin{pmatrix} D_0 \cos\alpha \\ D_0 \sin\alpha \exp(i\varphi) \end{pmatrix}$$

<u>Method 1:</u> Consider the components that are projected on the transmission axis of the analyzer A:



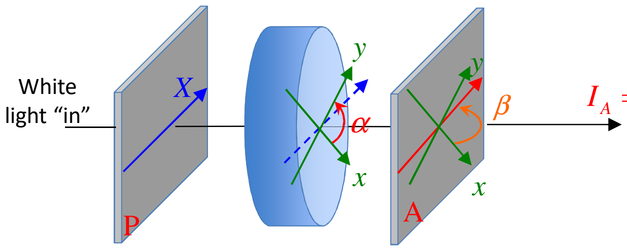
$$D_A = \qquad\qquad +$$

$$= \qquad\qquad +$$

$$= D_0 \cos\alpha \cos\beta + D_0 \sin\alpha \sin\beta \exp(i\varphi)$$

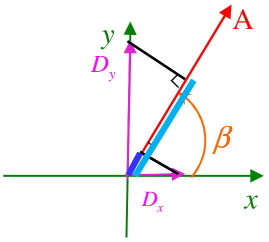
Next, find intensity $I_A = |D_A|^2$

Interference and retarders



$$I_A = |D_A|^2$$

$$I_A = |D_A|^2 = D_0^2\left[(\cos\alpha\cos\beta)^2 + (\sin\alpha\sin\beta)^2 + 2\cos\alpha\cos\beta\sin\alpha\sin\beta\cos\varphi\right]$$



Compare to:

$$I_{tot} = |D_{tot}|^2 = \left[|D_1|^2 + |D_2|^2 + 2D_1D_2\cos\varphi\right]$$

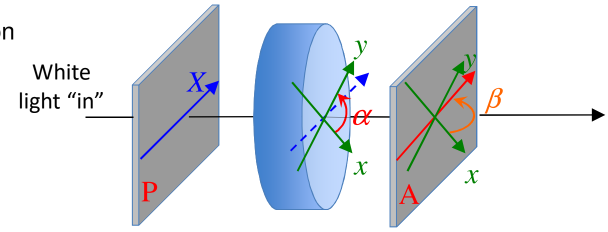
Equivalent to the interference between two rays with fields D_1 and D_2 which have a φ phase delay between them!

Interference and retarders

Method 2: Jones matrix notation

After retarder:

$$\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \begin{pmatrix} D_0 \cos\alpha \\ D_0 \sin\alpha \exp(i\varphi) \end{pmatrix}$$



Find transmission matrix for second polarizer A.
Recall:

$$\mathbf{P}_\theta = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$ or in the current case: $$\mathbf{P}_\beta = \begin{pmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{pmatrix}$$

$$\mathbf{D}_\mathbf{A} = \begin{pmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{pmatrix}\begin{pmatrix} D_0\cos\alpha \\ D_0\sin\alpha\exp(i\varphi) \end{pmatrix} = D_0\begin{pmatrix} [\cos\alpha\cos\beta + \sin\alpha\sin\beta\exp(i\varphi)]\cos\beta \\ [\cos\alpha\cos\beta + \sin\alpha\sin\beta\exp(i\varphi)]\sin\beta \end{pmatrix}$$

$$I_A = |D_{Ax}|^2 + |D_{Ay}|^2$$

Wave is indeed polarized linearly with an angle β to the axes of the retarder

$$I_A = D_0^2\left[(\cos\alpha\cos\beta)^2 + (\sin\alpha\sin\beta)^2 + 2\cos\alpha\cos\beta\sin\alpha\sin\beta\cos\varphi\right]$$

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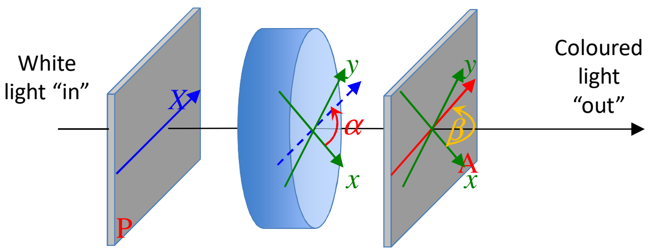
Interference and retarders



$$\varphi = \frac{2\pi}{\lambda}(n_e - n_o)d$$

$$I_A = D_0^2\left[(\cos\alpha\cos\beta)^2 + (\sin\alpha\sin\beta)^2 + 2\cos\alpha\cos\beta\sin\alpha\sin\beta\cos\varphi\right]$$

Result after sending the "output" through a prism:

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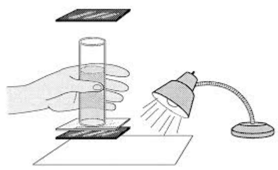
Lecture 8—Experiment



- Experiment:
  - Place a solution (corn syrup, sugar…) between two polarizers
  - Look at a light source through this "sandwich"
  - ⇒ observations
  - Rotate the nearest polarizer
  - ⇒ observations
  - Increase the amount of solution in the graduated cylinder
  - => observations

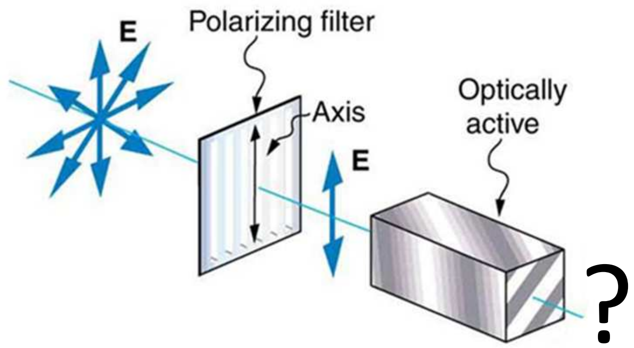
https://www.exploratorium.edu/snacks/rotating-light

Circular birefringence: Optical activity and the Faraday effect

Optical activity:

What is it?
Where does it come from?

Why care about it?



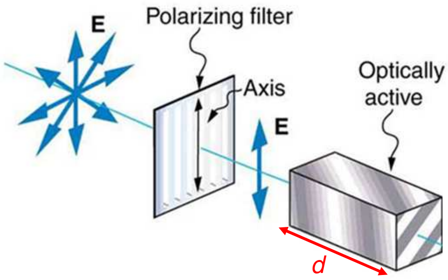
Optical activity or *circular* birefringence

- **Optical activity:** An optically active material will cause the **direction of incident linearly polarized light to rotate**, i.e., the polarization **remains linear** but the **orientation changes**.
- **The rotation angle depends on the distance traveled in the optically active medium**

$$\alpha = \rho\, d$$

ρ:
Depends on:

- material
- concentration
- temperature
- wavelength of light
- propagation direction



Why care about optical activity?

Exists in many molecules, in particular those of chemical and biological interest!

Examples:
Food and drugs

- Amino Acids
- Antibiotics
- Steroids
- Cocaine
- Diuretics
- Tranquilizers
- Analgesics
- Vitamins
- Carbohydrates
- Dextrose
- Lactose
- Fructose
- Sucrose
- Glucose

Examples: Flavor, Fragrance, and Essential Oil Industry

- Orange oil
- Citric acid
- Lavender oil
- Spearmint oil
- Lemon oil

RUDOLPH RESEARCH ANALYTICAL

Use the optical activity of substances in the food, drug and fragrance industries to quickly, cheaply, and non-destructively monitor quality, measure purity…

https://rudolphresearch.com/products/polarimeters/polarimetry-definitions/

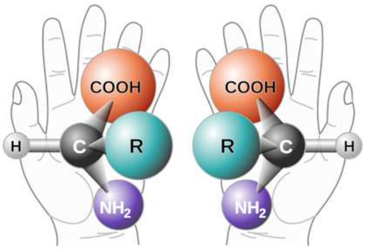
Where does optical activity come from?

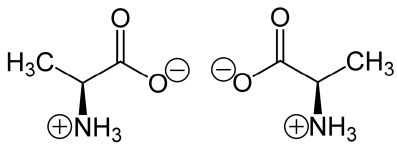
Optically active materials:

- Solutions containing **asymmetric molecules**
  - no symmetry plane, i.e.,

    such a molecule cannot be superimposed on its mirror image

Chiral



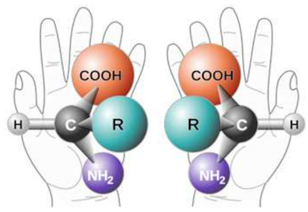


Amino acid

https://en.wikipedia.org/wiki/Chirality

Chirality: some vocabulary

Chiral molecules exist in general in two versions: a "left-hand" and "right-hand" version, **though often only one of the two may exist in living organisms.**



Enantiomorphs: a chiral object and its mirror image

→ same atoms and bond structure, but the geometrical positioning of atoms and functional groups in space differs so the molecules are mirror images of each other

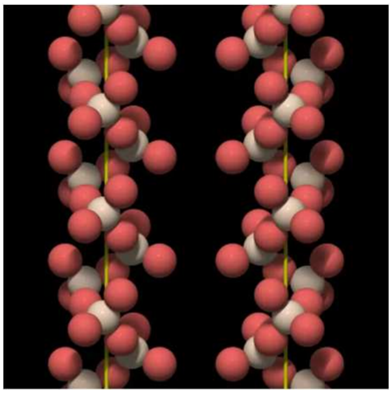
https://www.masterorganicchemistry.com/2018/09/10/types-of-isomers/

Where does optical activity come from?

Optically active materials:

- crystal of symmetric molecules **in an asymmetric arrangement**
- optical activity apparent for propagation along the optic axis

Quartz: symmetric molecules arranged in an asymmetric structure



http://www.quartzpage.de/gen_struct.html

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Rotation direction

$\alpha = \rho d$

The two enantiomorphs (or asymmetrical arrangements) give rise to a different **sign** for the parameter ρ

dextrorotatory clockwise for incoming light

levorotatory counter-clockwise for incoming light

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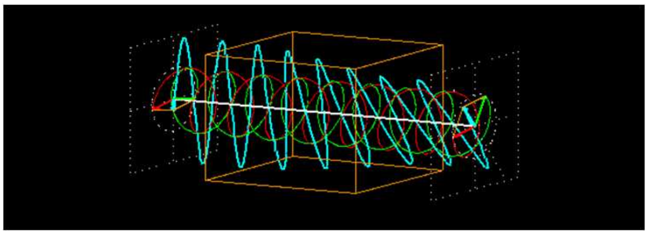
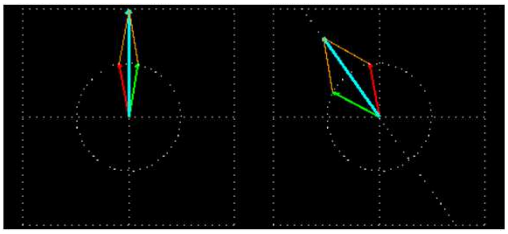
Towards Fresnel's interpretation of optical activity

Linearly polarized light may be considered the sum of **right-handed circularly** and **left-handed circularly** polarized light





http://cddemo.szialab.org/

Fresnel's phenomenological hypothesis

Fresnel: in an optically active material, the indices of refraction for right-handed and left-handed polarized light are different!

Recall for birefringent material: different indices for two orthogonal linear polarizations

$$k_R = \frac{2\pi}{\lambda} n_R \qquad k_L = \frac{2\pi}{\lambda} n_L$$

Using this hypothesis, define in the material:

Right circular polarization $\mathbf{E}_R = \frac{E_0}{2}\left[\cos(\omega t - k_R z)\hat{\mathbf{i}} - \sin(\omega t - k_R z)\hat{\mathbf{j}}\right]$

Left circular polarization $\mathbf{E}_L = \frac{E_0}{2}\left[\cos(\omega t - k_L z)\hat{\mathbf{i}} + \sin(\omega t - k_L z)\hat{\mathbf{j}}\right]$

⇩

Linear polarization outside of the material (e.g., in air)

In air: $k_R = k_L = k = \frac{2\pi}{\lambda}$

$$\mathbf{E} = \mathbf{E}_R + \mathbf{E}_L = E_0 \cos(\omega t - kz)\hat{\mathbf{i}}$$

Fresnel's phenomenological hypothesis

$$\mathbf{E}_R = \frac{E_0}{2}\left[\cos(\omega t - k_R z)\hat{\mathbf{i}} - \sin(\omega t - k_R z)\hat{\mathbf{j}}\right] \qquad \mathbf{E}_L = \frac{E_0}{2}\left[\cos(\omega t - k_L z)\hat{\mathbf{i}} + \sin(\omega t - k_L z)\hat{\mathbf{j}}\right]$$

$$k_R = \frac{2\pi}{\lambda} n_R \qquad k_L = \frac{2\pi}{\lambda} n_L$$

In the material, the initial linear polarization becomes: $\mathbf{E} = \mathbf{E}_R + \mathbf{E}_L$

$$\mathbf{E} = \frac{E_0}{2}\left[\cos(\omega t - k_R z)\hat{\mathbf{i}} - \sin(\omega t - k_R z)\hat{\mathbf{j}}\right] + \frac{E_0}{2}\left[\cos(\omega t - k_L z)\hat{\mathbf{i}} + \sin(\omega t - k_L z)\hat{\mathbf{j}}\right]$$

Using trigometric identities:

$$\cos a + \cos b = 2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

$$\sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

$$\mathbf{E} = E_0 \cos\left(\omega t - \left(\frac{k_R + k_L}{2}\right)z\right)\left[\cos\left(\frac{k_R - k_L}{2}\right)z\hat{\mathbf{i}} + \sin\left(\frac{k_R - k_L}{2}\right)z\hat{\mathbf{j}}\right]$$

$$\mathbf{E} = E_0 \cos\left(\omega t - \left(\frac{k_R + k_L}{2}\right)z\right)\begin{bmatrix}\cos\left(\frac{k_R - k_L}{2}\right)z \\ \sin\left(\frac{k_R - k_L}{2}\right)z\end{bmatrix}$$

Recall: $\mathbf{u}_\alpha = \begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}$

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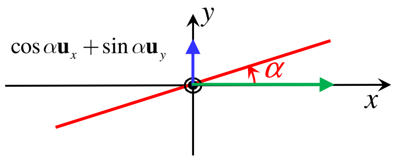
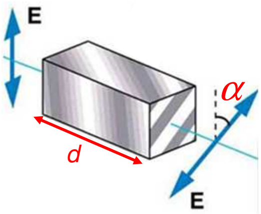
Fresnel's phenomenological hypothesis

$$\mathbf{E} = E_0 \cos\left(\omega t - \left(\frac{k_R + k_L}{2}\right)z\right)\begin{bmatrix}\cos\left(\frac{k_R - k_L}{2}\right)z \\ \sin\left(\frac{k_R - k_L}{2}\right)z\end{bmatrix}$$

Recall: $\mathbf{u}_\alpha = \begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}$

$\cos\alpha\,\mathbf{u}_x + \sin\alpha\,\mathbf{u}_y$





$$k_R = \frac{2\pi}{\lambda} n_R \qquad k_L = \frac{2\pi}{\lambda} n_L$$

For a material thickness *d*:

$$\mathbf{E}(z = d) = E_0 \cos\left(\omega t - \left(\frac{k_R + k_L}{2}\right)d\right)\begin{bmatrix}\cos\frac{\pi}{\lambda}(n_R - n_L)d \\ \sin\frac{\pi}{\lambda}(n_R - n_L)d\end{bmatrix}$$

$$\alpha = \frac{\pi}{\lambda}(n_R - n_L)d$$

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Fresnel's phenomenological hypothesis…using Jones vectors

- We know that a linear polarization will be rotated by an angle α.
- What is the Jones matrix for this? $M = R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

Apply this now to circularly polarized light:

Recall: $\mathbf{u}_L = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ +i \end{pmatrix}$ $\mathbf{u}_R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$M\mathbf{u}_{L,R} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\alpha \mp i\sin\alpha \\ \sin\alpha \pm i\cos\alpha \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(\cos\alpha \mp i\sin\alpha)\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \frac{1}{\sqrt{2}} e^{\mp i\alpha}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = e^{\mp i\alpha}\mathbf{u}_{L,R}$$

Circularly polarized waves are eigenmodes of an optically active medium!

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The parameter "ρ" and the indices of refraction

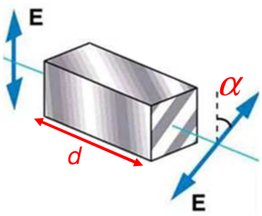
$$M\mathbf{u}_{L,R} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \frac{1}{\sqrt{2}} e^{\mp i\alpha}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} = e^{\mp i\alpha}\mathbf{u}_{L,R}$$

Phase difference between left and right circularly polarized waves after passing through an optically active material of thickness *d*

$$\alpha - (-\alpha) = 2\alpha = (k_R - k_L)d = \frac{2\pi}{\lambda}(n_R - n_L)d$$

$$\alpha = \frac{\pi}{\lambda}(n_R - n_L)d$$

$\alpha = \rho d$ ➡ $$\rho = \pm\frac{\pi}{\lambda}(n_R - n_L)$$



Caution! Sign convention different in French and English!!

$\rho > 0$ dextrorotatory in English; lévogyre en français!!!

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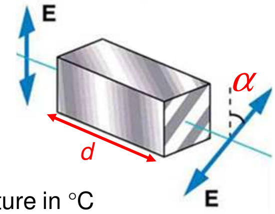
Specific rotation



Recall: $\alpha = \rho d$

Specific rotation: $[\alpha]_\lambda^T = \frac{\alpha}{\gamma d}$

T: temperature in °C

λ: wavelength

α: measured rotation angle

γ: solution concentration in g/l

D=>sodium line 589.3 nm

Compound	$[\alpha]_D$ [°/(dm g/L)]	Compound	$[\alpha]_D$ [°/(dm g/L)]
Benzene	0.00	Cholesterol	-31.50
α-D-glucose	+112.00	Morphine	-132.00
β-D-glucose	+18.70	Penicillin V	+223.00
Camphor	+44.26	Sucrose	+66.47

+: dextrorotatory

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Wavelength dependence of the rotatory power

- Optical activity depends strongly on the wavelength:
- Biot phenomenological law: $\rho = \frac{C}{\lambda^2}$
- If you take into account the wavelengths λ_i of the electronic transitions that exist in the UV: $\rho = \sum_i \frac{c_i}{\lambda^2 - \lambda_i^2}$
- Seems like the optical rotatory power may be related in some way to the index of refraction! (looks like the Lorentz model)

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60. Specific Rotation

Table 60-1. SPECIFIC ROTATION*

Solids

Substance	Wave-length, μm	Rota-tion, deg/min	Substance	Wave-length, μm	Rota-tion, deg/min
Cinnabar (HgS)........	*D*	+32.5	Quartz..........	0.3726	+58.894
Lead hyposulfate.......	*D*	5.5		0.3609	63.628
Potassium hyposulfate..	*D*	8.4		0.3582	64.459
Quartz...............	3.676	0.34		0.3466	69.454
	1.342	3.89		0.3441	70.587
	0.7604	12.668		0.3402	72.448
	0.7184	14.304		0.3360	74.571
	0.6867	15.746		0.3286	78.579
	0.6562	17.318		0.3247	80.459
	0.5895932	21.7010		0.3180	84.972
	0.5895	21.684		0.2747	121.052
	0.5892617	21.729		0.2571	143.266
	0.5889965	21.7492		0.2313	190.426
	0.5889	21.727		0.2265	201.824
	0.5460741	25.538		0.2194	220.731
	0.5269	27.543		0.21740	229.96
	0.4861	32.773		0.2143	235.972
	0.4307	42.604		0.1750	453.5
	0.4101	47.481		0.1525	776.0
	0.3968	51.193	Sodium bromate..	*D*	2.8
	0.3933	52.155	Sodium chlorate..	*D*	3.13
	0.3820	55.625			

Specific rotation or rotatory power is given in degrees per decimeter for liquids and solutions and in degrees per millimeter for solids; + signifies right-handed rotation, − left. Specific rotation varies with the wavelength of light used, with temperature and, in the case of solutions, with the concentration. When sodium light is used, indicated by *D* in the wavelength column, a value of $\lambda = 0.5893$ may be assumed.

Optical rotatory power for a large number of organic compounds will be found in the "International Critical Tables," vol. VII; for sugars, vol. II.

*Most of the data taken from "Handbook of Chemistry and Physics," 36th ed., pp. 2752, 2753, 2754, Chemical Rubber Publishing Company, 1954–1955.

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A microscopic theory for optical activity

Recall Lecture 3:

•Material constants depend on **E**-field in the past (i.e., the response is non-instantaneous, i.e., **not** localized in *time*)

$$D_j(\mathbf{r},t) = \sum_l \int dt' \varepsilon_{jl}(t-t') E_l(\mathbf{r},t')$$

frequency dispersion, material properties depend on ω

frequency dispersion ⇔ absorption

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A microscopic theory for optical activity

•Material constants depend on **E**-field *within a small volume*, e.g., the volume of a molecule (i.e., response is non-local in ***space***)

$$D_j(\mathbf{r},\omega) = \sum_l \frac{1}{V} \int_V d^3\mathbf{r}' \, \varepsilon_{jl}(\mathbf{r}-\mathbf{r}',\omega) \, E_l(\mathbf{r}',\omega)$$

wavevector dispersion, material properties depend on k, i.e., optical rotation varies with wavelength and **direction**

Wavevector dispersion (**optical rotation dispersion**) ⇔ ***circular* dichroism** (i.e., differential absorption of left and right-handed light)

Can demonstrate that **spatial dispersion** gives rise to **optical activity**

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Conclusion: natural optical activity

- Spatial dispersion leads to circular birefringence!
- Right-handed and left-handed circularly polarized waves are the eigenstates of these materials!!!
- Different indices of refraction for right-handed and left-handed circularly polarized waves!

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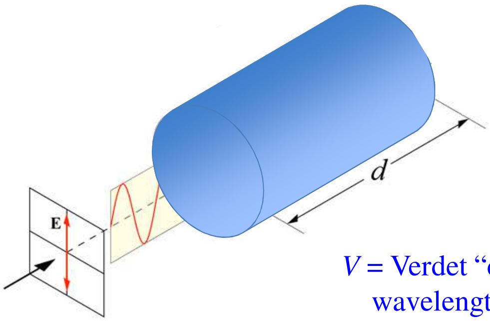
Faraday effect—<u>induced</u> circular birefringence

Occurs in most transparent dielectric materials, i.e., <u>isotropic / symmetrical</u> structures

Applied magnetic field parallel to the propagation direction



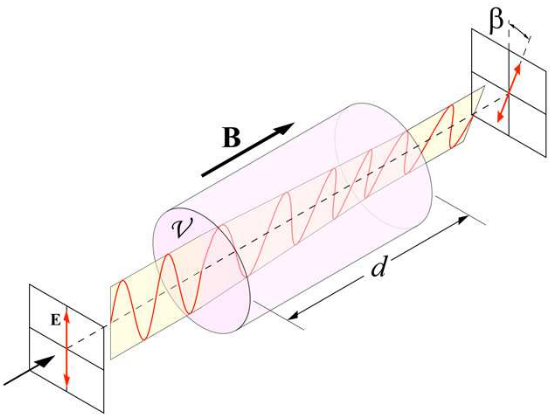
$$\beta = VBd$$

V = Verdet "constant" (varies with wavelength and temperature)

Faraday effect—<u>induced</u> circular birefringence

Important difference with optical activity: it is the ***<u>magnetic field direction</u>*** and **NOT** the propagation direction which determines the rotation direction (for a given V)

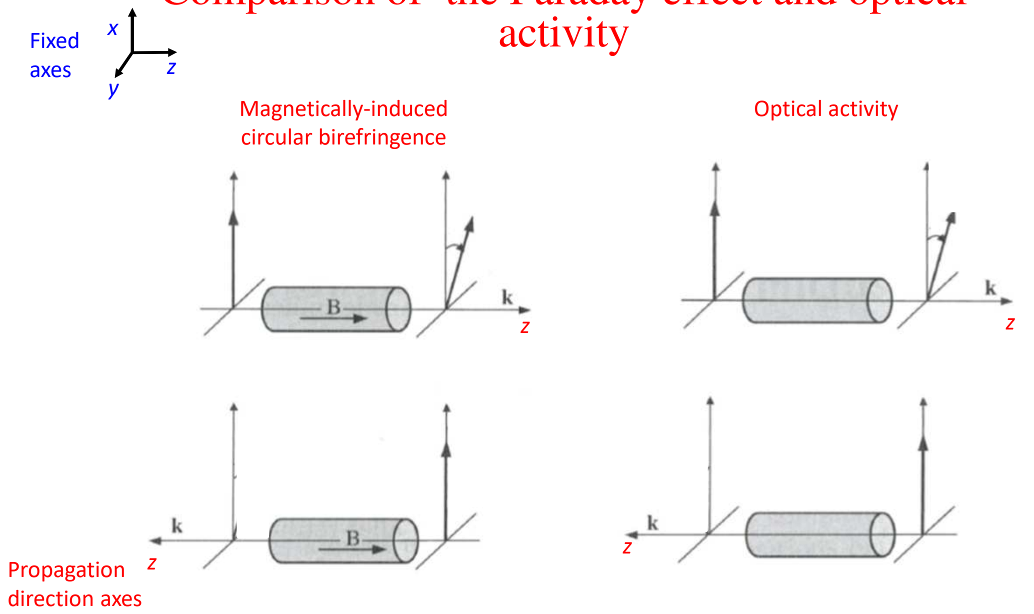
V = Verdet "constant" : same sign in English and French



In this case V>0

$$\beta = VBd$$

Comparison of the Faraday effect and optical activity



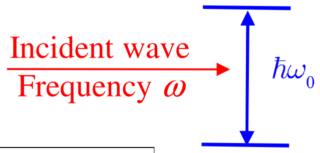
Reciprocity

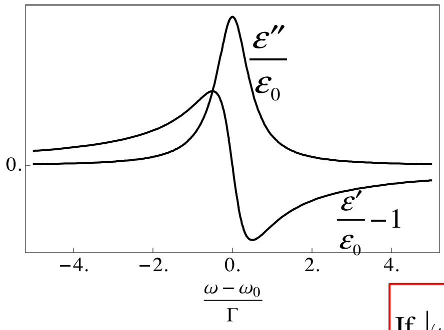
Compare the polarization rotation for propagation in the +z and –z directions

Rotation angle measurement	Optical activity α	Faraday effect β
With respect to absolute (x,y,z) coordinates		
With respect to the propagation direction **k** Polarization eigenmodes		
	Reciprocal effect	Non-reciprocal effect (symmetry broken by **B** field)

Recall: Lorentz model

Atomic resonances at frequences ω_0





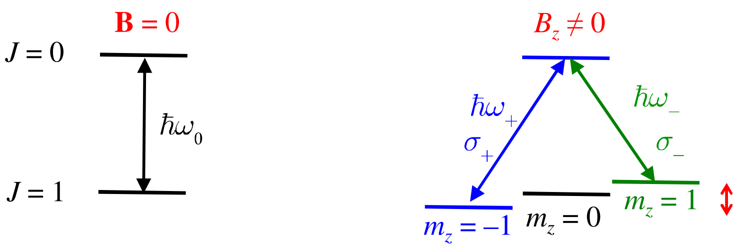
$$\frac{\varepsilon'(\omega)}{\varepsilon_0} = 1 + \frac{\omega_p^2\left(\omega_0^2 - \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2}$$

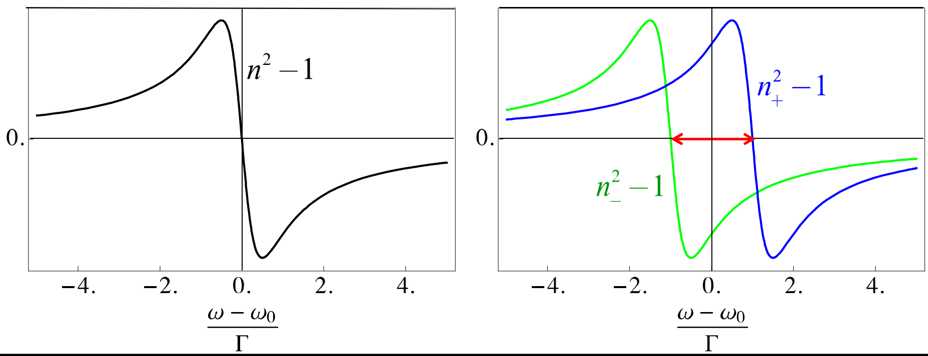
$$\frac{\varepsilon''(\omega)}{\varepsilon_0} = \frac{\omega_p^2\omega\Gamma}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2}$$

If $|\omega - \omega_0| \ll \omega_0 : n^2 = 1 + \frac{K(\omega_0 - \omega)}{(\omega - \omega_0)^2 + \Gamma^2/4}$

Zeeman effect

The ground level is degenerate. Light is propagating in the z direction

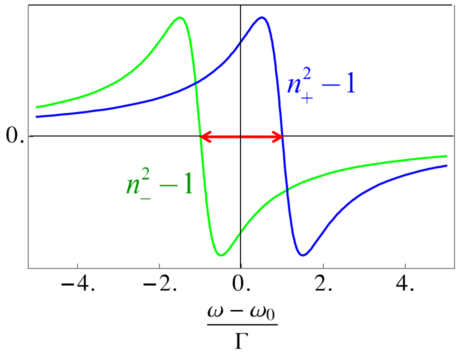




Zeeman effect and the Faraday effect

The ground level is degenerate for B=0. Light is propagating in the z direction

With an applied field:

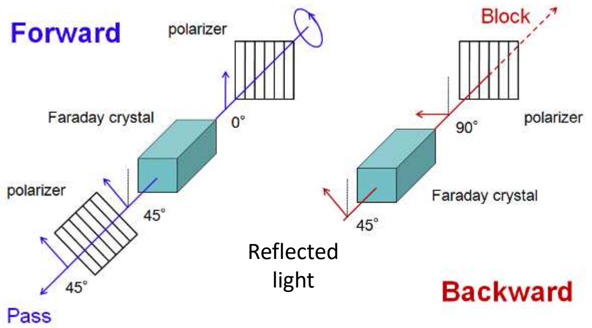


$n_+ \neq n_-$

Circular birefringence

Rotation of linear polarization

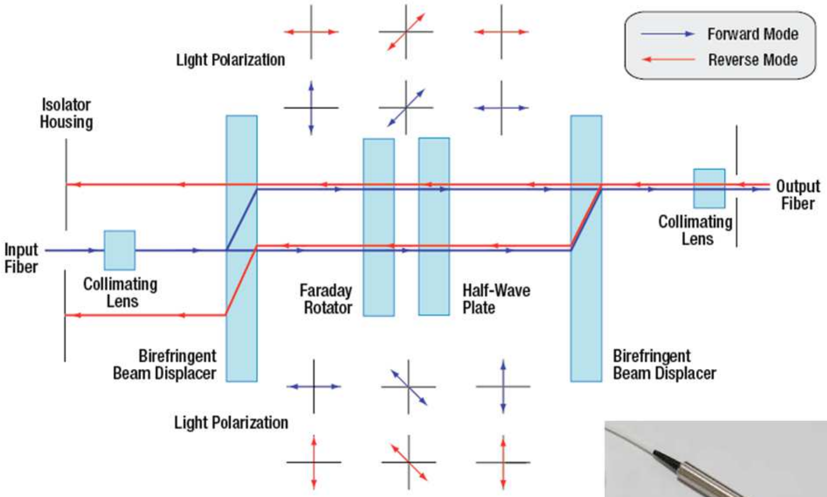
Application: optical isolator



No reflected light!

http://www.chinacablesbuy.com/what-is-fiber-optic-isolator.html

Polarization-independent optical isolator



Use circular and linear birefringent materials together!

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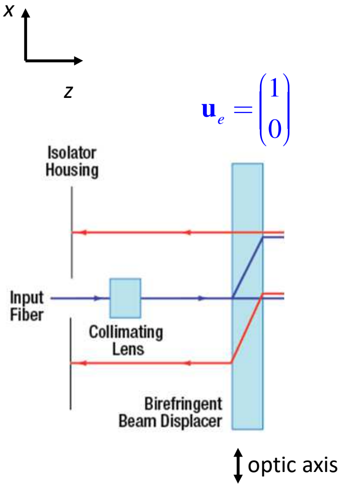
Polarization-independent optical isolator



$$\mathbf{u}_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{u}_o = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Next: use Faraday rotator to rotate polarization by 45°

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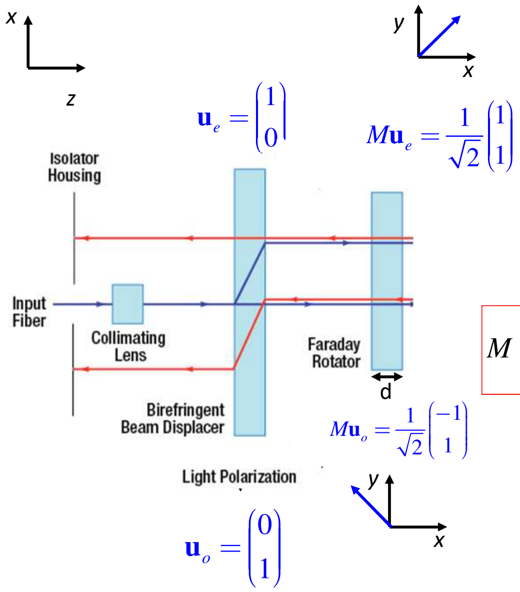
Polarization-independent optical isolator



Next: use Faraday rotator to rotate polarization by 45°

$$\boxed{\beta = VBd}$$

Choose *V*, *B*, *d* so that $\beta = \frac{\pi}{4}$

$$M = R(\alpha) = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$$

$$M = R\left(\alpha = \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

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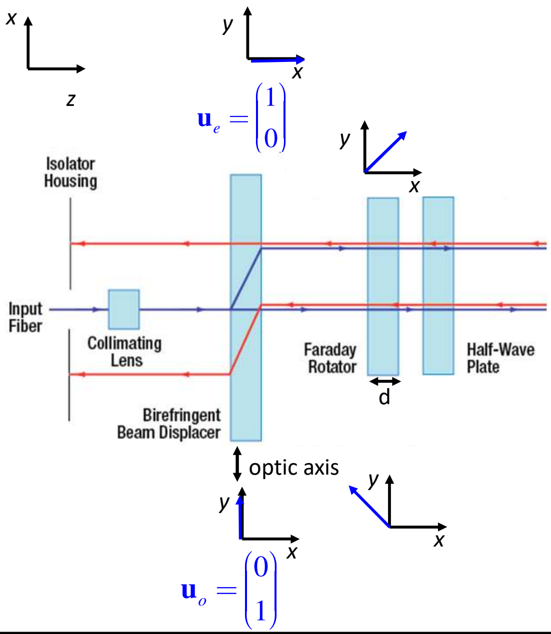
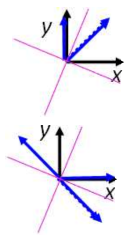
Polarization-independent optical isolator



Next: use half wave plate to rotate another 45° so that polarization is perpendicular to the result after the beam displacer. How should the half wave plate be oriented?



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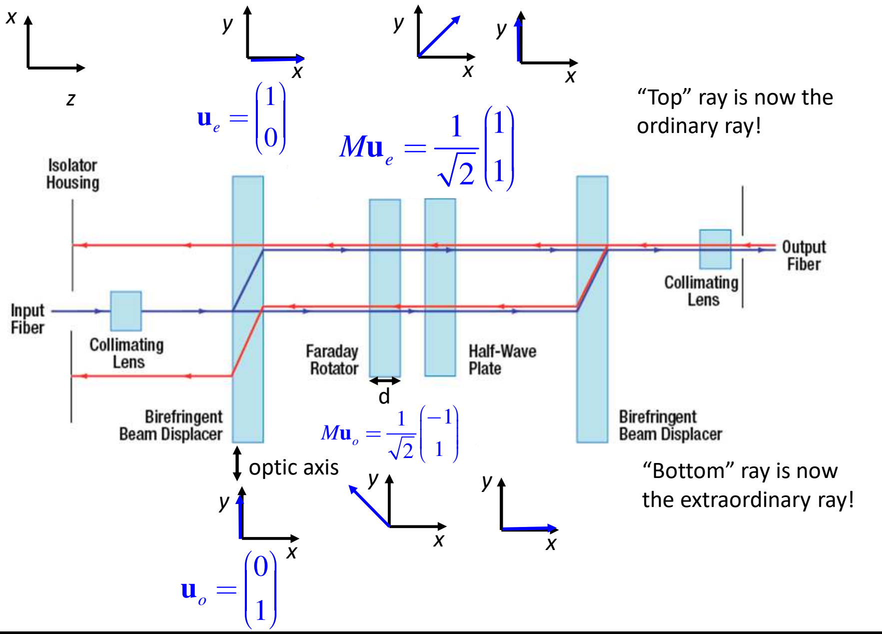
Polarization-independent optical isolator



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François Arago (1786-1853)

Conclusion and summary

Jean-Baptiste Biot (1774-1862)
here with Gay-Lussac

Louis Pasteur (1822-1895)

Circular birefringence:

- Linear polarization remains linear but is rotated!
- May be intrinsic to the material or may be induced
- Optical activity:
  - Found in non-centrosymmetric materials
  - May be due to the arrangement of symmetric molecules or due the asymmetry of the molecules themselves
  - Characteristic of many biological molecules
  - Reciprocal effect
- Faraday effect:
  - Induced by a magnetic field
  - Non reciprocal

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