TP marche aléatoire et théorème de la limite centrale

A. Loi binomiale

1/ une réalisation : ○●●○○●●○ proba $p^n q^{M-n}$

p q q p p q q p

d'où $\boxed{\Pi_M(n) = \binom{M}{n} p^n q^{M-n}} = \boxed{\binom{M}{n} p^n (1-p)^{M-n}}$ $\quad \sum_{n=0}^{M} \Pi_M(n) = (p+q)^M = 1$

2/ $\langle n^k \rangle = \sum_{n=0}^{M} \Pi_M(n)\, n^k$, on peut utiliser $n\binom{M}{n} = n\dfrac{M!}{n!(M-n)!}$

méthode itérative → fastidieux $\quad = M \dfrac{(M-1)!}{(n-1)!(M-1-(n-1))!}$

$= M\binom{M-1}{n-1}$

3/ $G_M(s) = \langle s^n \rangle$; cours $s = e^{-\beta}$

a) $\dfrac{d^k G}{ds^k} = \langle n(n-1)\cdots(n-k+1)\, s^{n-k} \rangle$

so for $s=1$ $\quad \left.\dfrac{d^k G}{ds^k}\right|_{s=1} = \langle n(n-1)(n-2)\cdots(n-k+1) \rangle$ is a function of the momen

first two : $\quad \left.\dfrac{dG}{ds}\right|_{s=1} = \langle n \rangle$; $\quad \left.\dfrac{d^2G}{ds^2}\right|_{s=1} = \langle n^2 \rangle - \langle n \rangle$; ...

b) $G_M(s) = \sum_{n=0}^{M} \Pi_M(n)\, s^n = \sum_{n=0}^{M} \binom{M}{n} (ps)^n (1-p)^{M-n} = (1-p+ps)^M$

so $\boxed{G_M(s) = (1+p(s-1))^M}$ simple !

$\underline{\langle n \rangle} = pM(1+p(1-1))^{M-1} = \underline{Mp}$; $\langle n^2 \rangle - \langle n \rangle = pM\, p(M-1)$

$\quad \overset{''}{Mp} \quad = (pM)^2 - Mp^2$

so $\underline{Var(n)} = \langle n^2 \rangle - \langle n \rangle^2 = (pM)^2 + Mp - Mp^2 - (pM)^2 = \underline{Mp(1-p)}$

$\underline{\dfrac{\sigma_n}{\langle n \rangle} = \dfrac{1}{\sqrt{M}}\sqrt{\dfrac{1-p}{p}}}$

c) $W_M(\beta) = -\ln G_M(e^{-\beta}) = -\ln\left(1+p(e^{-\beta}-1)\right)^M$

$\downarrow$ cours $\quad = -M\ln\left(1+p\left(\cancel{1}-\beta+\frac{\beta^2}{2}-\cancel{1}\right)\right) \quad \ln(1+x) = x$

$\beta\langle n \rangle - \dfrac{\beta^2}{2} Var(n) \quad = -M\left(p\left(-\beta+\frac{\beta^2}{2}\right) - \frac{1}{2}(-\beta p)^2\right)$

$\downarrow$

identification $\quad = Mp\beta + -\dfrac{Mp}{2}(1-p)\beta^2 + \ldots$

4/ $\ln \Pi_M(n) = \ln M! - \ln n! - \ln(M-n)! + n \ln p + (M-n)\ln q \quad ; q = 1-p$

on a $\ln n! \simeq n\ln n - n$, $n \gg 1$, $\dfrac{d \ln n!}{dn} \simeq \ln n$ $\Big\}$ d'où $\dfrac{d \ln \Pi_M(n)}{dn} = -\ln n - (-1)\ln(M-n) + \ln p - \ln q$

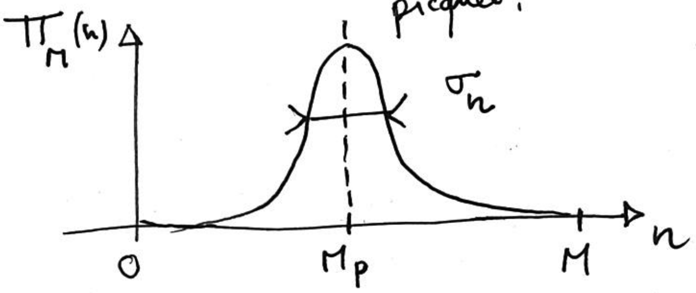
$= 0 \Rightarrow \ln\left(\dfrac{n^*}{M-n^*}\right) = \ln \dfrac{p}{q}$

soit $\dfrac{n^*}{M-n^*} = \dfrac{p}{q} \Leftrightarrow n^*(\overbrace{q+p}^{=1}) = Mp \Rightarrow \boxed{n^* = Mp = \langle n \rangle}$

dérivée seconde: $\left.\dfrac{d^2 \ln \Pi_M(n)}{dn^2}\right|_{n^*} = -\dfrac{1}{n^*} - \dfrac{1}{M-n^*} = -\dfrac{M - Mp + Mp}{Mp(M-Mp)} = -\dfrac{1}{\sigma_n^2}$

donc, on a $\ln \Pi_M(n) \underset{\substack{n \gg 1 \\ n \to n^*}}{\simeq} \ln \Pi_M(n^*) - \dfrac{(n-n^*)^2}{2\sigma_n^2} + \dots$

soit $\boxed{\Pi_M(n) \simeq A\, e^{-\frac{(n-\langle n \rangle)^2}{2\sigma_n^2}}} \xrightarrow[\text{normalisation}]{} A = \dfrac{1}{\sqrt{2\pi\sigma_n^2}}$



gaussienne de largeur $\sigma_n = \sqrt{M}\sqrt{p(1-p)} \ll \langle n \rangle$

B. Marcheur

1/ $x = na + (M-n)(-a) = (2n-M)a$

$\langle x \rangle = (2\langle n \rangle - M)a = (2p-1)Ma = (p-q)Ma$

$\mathrm{Var}(x) = (2a)^2\, \mathrm{Var}(n) = 4a^2 p(1-p)M$

2/ $\boxed{V = (p-q)\dfrac{a}{\tau}}$ car $M = t/\tau$

3/ $\boxed{D = \dfrac{\mathrm{Var}(x)}{2M\tau} = 2p(1-p)\dfrac{a^2}{\tau}}$

4/ $dx = 2a\, dn$; $P_t(x)\,dx \underset{n \gg 1}{\simeq} \Pi_M(n) \underbrace{dn}_{=1} \longrightarrow P_t(x) \simeq \dfrac{1}{2a} \Pi_{M=t/\tau}\left(\dfrac{1}{2}\left[M + \dfrac{x}{a}\right]\right)$ → gaussienne

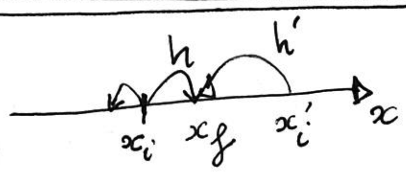
$n = \dfrac{1}{2}\left[M + \dfrac{x}{a}\right]$

$\simeq \dfrac{1}{\sqrt{2\pi\sigma_{x(t)}^2}}\, e^{-\frac{1}{2}\left(\frac{x - Vt}{\sigma_x(t)}\right)^2}$; $\sigma_x^2 = 2D$

pour $p = \frac{1}{2}$, $V = 0$, $D = \dfrac{a^2}{2\tau}$; $\boxed{P_t(x) \simeq \dfrac{1}{\sqrt{4\pi D t}}\, e^{-\frac{x^2}{4Dt}}}$ → $\langle x^2 \rangle = 2Dt$ diffusion

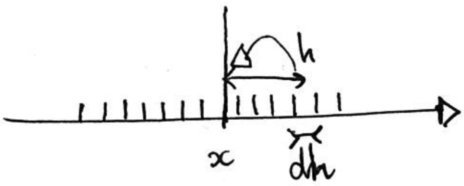
C. Distribution continue de sauts et universalité

1/



$$\mathrm{Proba}[x_f] = \underbrace{\mathrm{Proba}[x_i(h)]_t}_{\text{être en } x_i} \overset{\boxed{\text{et}}}{\times} \underbrace{\overbrace{p(h)dh}^{\text{Proba de saut de } h}}_{\text{faire un saut de } h}$$

$\boxed{\text{ou}} \rightarrow +$ $\mathrm{Proba}[x_i(h+dh)]_t \times p(h+dh)dh$



$\boxed{\text{ou}} \rightarrow +$

$\vdots$

$$\mathrm{Proba}[x, t+\tau] = \int dh\, p(h)\, \mathrm{Proba}[x-h, t]$$

en simplifiant par dx : $\boxed{P_{t+\tau}(x) = \int dh\, p(h)\, P_t(x-h)} = (p * P_t)(x)$

$\rightarrow$ produit de convolution. $*$

2/ convolution :

$$P_{t+M\tau}(x) = (\underbrace{p*p*\dots*p}_{M \text{ fois}} * P_t)(x)$$

$\rightarrow$ distribution initiale $P_0(x) = \delta(x)$

produit de convolution de deux gaussiennes : $\boxed{\sigma_{f\otimes g}^2 = \sigma_f^2 + \sigma_g^2 \,;\, \mu_{f\otimes g} = \mu_f + \mu_g}$

$$(g*g)(x) = \int_{-\infty}^{+\infty} dh\, \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{h^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-h)^2}{2\sigma^2}}$$

$$= \frac{e^{-\frac{x^2}{2\sigma^2}}}{(\sqrt{2\pi\sigma^2})^2} \underbrace{\int_{-\infty}^{+\infty} dh\, e^{-\frac{1}{2\sigma^2}(2h^2 - 2hx)}}_{= \int_{-\infty}^{+\infty} dh\, e^{-\frac{(h-\cancel{x})^2}{\sigma^2}} e^{+\frac{x^2}{4\sigma^2}}}$$

$$= \frac{e^{-\frac{x^2}{4\sigma^2}}}{(\sqrt{2\pi\sigma^2})^2} \sqrt{2\pi\frac{\sigma^2}{2}} = \frac{e^{-\frac{x^2}{2(2\sigma^2)}}}{\sqrt{2\pi(2\sigma^2)}}$$

solution for $P_0(x) = \delta(x)$:

$$P_{M\tau}(x) = \frac{1}{\sqrt{2\pi\sigma_M^2}} e^{-\frac{x^2}{2\sigma_M^2}} \quad \text{avec} \quad \boxed{\sigma_M^2 = M\sigma^2}$$

Méthode 2.

using $\int dh\, p(h) = 1$

3/ a)
$$\left.\begin{array}{l} P_{t+\tau}(x) \simeq P_t(x) + \tau \dfrac{\partial P_t}{\partial t} \\ P_t(x-h) \simeq P_t(x) - h\dfrac{\partial P_t}{\partial x} + \dfrac{h^2}{2}\dfrac{\partial^2 P_t}{\partial x^2} \end{array}\right\} \tau \partial_t P_t \simeq \int dh\, p(h)\left(-h\partial_x P_t + \frac{h^2}{2}\partial^2_{x^2} P_t\right)$$

$$\simeq -\underbrace{\left(\int dh\, p(h) h\right)}_{\langle h\rangle}\partial_x P_t + \frac{1}{2}\underbrace{\left(\int dh\, h^2 p(h)\right)}_{\langle h^2\rangle}\partial^2_{x^2} P_t$$

d'où
$$\boxed{\tau \partial_t P_t(x) = -\langle h\rangle \partial_x P_t + \frac{1}{2}\langle h^2\rangle \partial^2_{x^2} P_t}$$

b) $\tau \sim \varepsilon$, $\langle h\rangle \sim \langle h^2\rangle \sim \varepsilon$ soit $\sigma_h^2 = \underbrace{\langle h^2\rangle}_{\sim \varepsilon} - \underbrace{\langle h\rangle^2}_{\sim \varepsilon^2} \simeq \langle h^2\rangle$

posons $V = \dfrac{\langle h\rangle}{\tau}$ $D = \dfrac{\langle h^2\rangle}{2\tau}$ alors

$$\boxed{\partial_t P = -V\partial_x P + D\partial^2_{xx} P}$$

c) solution en posant $X = x - Vt$; $T = t$; $T = t$, $x = X + VT$

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial}{\partial t}\underbrace{\frac{\partial t}{\partial X}}_{=0}$$

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial x}\underbrace{\frac{\partial x}{\partial T}}_{V} + \frac{\partial}{\partial t}\underbrace{\frac{\partial t}{\partial T}}_{=1} = \frac{\partial}{\partial t} + V\frac{\partial}{\partial x}$$

puis TF sur l'espace.

Solution:
$$\boxed{P(x,t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{(x-Vt)^2}{4Dt}}}$$

D. to infinity and beyond

1/ $p(\vec{h}) = p(h_1)\times p(h_2)\times\dots\times p(h_d)$ → gaussiennes

soit $\boxed{P_t(\vec{x}) = P_t(x_1)\times\dots\times P_t(x_d)}$ si $P_0(\vec{x}) = \delta(x_1)\dots\delta(x_d)$

les x_i sont <u>indépendantes</u>

2/ $\langle \vec{x}^2\rangle = \left\langle \left(\sum_{i=1}^{d} x_i \vec{e}_i\right)^2\right\rangle = \left\langle \sum_{i=1}^{d} x_i^2 + \sum_{i\neq j} x_i x_j\right\rangle = \sum_{i=1}^{d}\langle x_i^2\rangle + \sum_{i\neq j}\langle x_i x_j\rangle$

or pour deux variables indépendantes:

$$\langle x_i x_j\rangle = \int dx_i\, dx_j \underbrace{P_t(x_i)\,P_t(x_j)}_{\text{loi jointe}}\, x_i x_j = \left(\int dx_i\, P_t(x_i)\,x_i\right)\left(\int dx_j\, P_t(x_j)\,x_j\right)$$

$$\boxed{\langle x_i x_j\rangle = \langle x_i\rangle\langle x_j\rangle}$$

↑ indépendance

ici, par symétrie $\begin{cases}\langle x_i^2\rangle = \langle x_j^2\rangle \\ \langle x_i\rangle = 0\end{cases}$, d'où

$$\boxed{\langle \vec{x}^2\rangle = d\,\langle x_i^2\rangle = 2dDt}$$

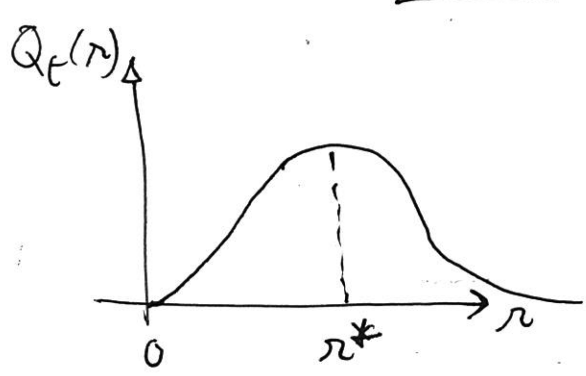
3/ <u>loi marginale à 2d:</u>

deux étapes : (i) chgt de variables
(ii) loi marginale

on écrit (i) $p(r,\theta)\,dr\,d\theta = P_t(x,y)\,dx\,dy$ et $dx\,dy = r\,dr\,d\theta$

soit $p(r,\theta) = r\,P_t(x(r,\theta), y(r,\theta)) = r\,\frac{1}{4\pi Dt}\,e^{-\frac{r^2}{4Dt}}$

puis (ii) $\boxed{Q_t(r) = \int_0^{2\pi} d\theta\; p(r,\theta) = \frac{r}{2Dt}\,e^{-r^2/4Dt}}$



<u>calcul:</u> $r^* = \sqrt{2Dt}$

$\langle r\rangle = \sqrt{\pi Dt}$

$\langle r^2\rangle = 4Dt$