Student Self-Assessment
Nonlinear wave equation & Nonlinear susceptibility tensors

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The electric field amplitude of a wave at ω_j, which propagates along the direction (Oz), is described by the relation : $\boldsymbol{\mathcal{E}}_j(z,t) = \boldsymbol{E}(\omega_j)e^{-\imath\omega_j t} + C.C.$, with $\boldsymbol{E}(\omega_j) = A_j(z)\, e^{\imath k(\omega_j)z}\boldsymbol{e}_j$. The related field intensity is given by: $I_j = 2n(\omega_j)c\,\epsilon_0|A_j(z)|^2$.

EXERCISE 1

A monochromatic electromagnetic wave is propagating in a nonlinear and homogenous medium. Its complex amplitude $\boldsymbol{E}(\omega)$ satisfies the wave equation:

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{E}(\omega) = \frac{\omega^2}{c^2}\underline{\underline{\epsilon}}(\omega)\boldsymbol{E}(\omega) + \omega^2\mu_0\boldsymbol{P}_{NL}(\omega). \tag{1}$$

One reminds the relation: $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{E}) - \Delta\boldsymbol{E}$. In the following, the specific case of an **isotropic medium** will be considered.

1. What can you say about $\underline{\underline{\epsilon}}(\omega)$ and the term $\boldsymbol{\nabla}.\boldsymbol{E}$? Justify your answer.

2. Assuming a wave propagating along the z direction, rewrite Eq. (1).

3. Expressing the complex amplitude as $\boldsymbol{E} = A(z)e^{ikz}\boldsymbol{e}$, show that the wave equation is reduced to a first order differential equation in z. Explicit any approximation(s).

EXERCISE 2

You are in charge of a new project in your company: the development of a laser source at 532 nm based on the PicoYAG laser module, already available in your product catalogue. The PicoYAG delivers sub-ns pulse duration at 1064 nm with a 1 kW peak-power.

You seek to evaluate an order of magnitude for the second-harmonic generation (SHG) efficiency throughout a 2 mm long lithium niobate ($LiNbO_3$) crystal. The refractive indices at 1064 nm and 532 nm are respectively equal to $n_\omega = 2.155$ and $n_{2\omega} = 2.234$. In the following, the effective nonlinear susceptibility is taken equal to 18 pm/V. The beam diameter inside the crystal is kept equal to 170 μm for both beams at ω and 2ω. Subsequently, the undepleted pump approximation is assumed.

1. Derive the phase matching condition. Can it be fulfilled?

2. Show that the intensity at 2ω follows

$$I_{2\omega,z} = \frac{(2\omega)^2}{2n^3c^3\varepsilon_0}\left|\chi^{(2)}_{eff}\right|^2 \sin^2\left(\frac{\Delta k}{2}z\right)\frac{I_\omega^2}{\Delta k^2} \tag{2}$$

Comment on the evolution of $I_{2\omega,z}$ for various Δk situations.

3. For $\Delta k \neq 0$, show that the maximum SHG efficiency is reached for $L_c = \pi/\Delta k$, the coherence length. Calculate L_c and the maximum expected SHG efficiency.

4. The SHG efficiency relation (2) can be rewritten :

$$\eta_{\mathrm{SHG}} = \frac{I_{2\omega}}{I_\omega} = \frac{(2\omega)^2}{8\epsilon_0 n_\omega^2 n_{2\omega} c^3} \left|\chi_{\mathrm{eff}}^{(2)}\right|^2 \operatorname{sinc}^2(\Delta k z/2)\; I_\omega\; z^2.$$

   Assuming a perfect phase matching condition, calculate the expected SHG efficiency. Comment about the validity of the undepleted pump approximation.

$\varepsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}