NONLINEAR OPTICS

Ch. 5 3rd ORDER NONLINEARITIES

I. **Optical Kerr Effect :** self-focusing, nonlinear Shrödinger equation, self-phase modulation, solitons

II. **Four-wave Mixing**

III. **Raman Scattering :** spontaneous and stimulated Raman scattering, Raman amplification, Raman laser

IV. **Brillouin Scattering :** spontaneous and stimulated Brillouin scattering

Energy transfer between a wave and the medium

Time-averaged power density transferred from the EM wave to the medium : $-\frac{\partial W}{\partial t} = \langle \vec{E}.\frac{\partial \vec{P}}{\partial t} \rangle$

$$\vec{E}(t) = \vec{E}(\omega)e^{-i\omega t} + CC$$
$$\vec{P}(t) = \vec{P}(\omega)e^{-i\omega t} + CC$$

$$\langle \vec{E}.\frac{\partial \vec{P}}{\partial t} \rangle = -i\omega[\vec{E}(-\omega).\vec{P}(\omega) - \vec{E}(\omega).\vec{P}(-\omega)]$$

$$\vec{P}(\omega) = \varepsilon_0 \underline{\underline{\underline{\chi}}}^{(3)}(\omega_P, -\omega_P, \omega)\vec{e}_P\vec{e}_P\vec{e}\ |E(\omega_P)|^2 E(\omega)$$

$$\langle \vec{E}.\frac{\partial \vec{P}}{\partial t} \rangle = -i\omega\varepsilon_0\left[\vec{e}.\underline{\underline{\underline{\chi}}}^{(3)}(\omega_P, -\omega_P, \omega)\vec{e}_P\vec{e}_P\vec{e} - \vec{e}.\underline{\underline{\underline{\chi}}}^{(3)*}(\omega_P, -\omega_P, \omega)\vec{e}_P\vec{e}_P\vec{e}\right]|E(\omega_P)|^2|E(\omega)|^2$$

$$\chi^{(3)} = \underbrace{\chi^{(3)'}}_{\text{Real part}} + i\underbrace{\chi^{(3)''}}_{\text{Imaginary part}}$$

$$\boxed{\langle \vec{E}.\frac{\partial \vec{P}}{\partial t} \rangle = 2\omega\varepsilon_0\ \vec{e}.\underline{\underline{\underline{\chi}}}^{(3)''}(\omega_P, -\omega_P, \omega)\vec{e}_P\vec{e}_P\vec{e}\ |E(\omega_P)|^2|E(\omega)|^2}$$

Conclusion :
If $\chi^{(3)}$ **is purely real** = **NO energy transfer** between the wave and the material

I- Optical Kerr Effect

- ***Nonlinear refractive index***

$$\vec{E}(t)=\vec{E}(\omega)e^{-i\omega t}+CC \qquad \vec{P}_L(\omega)=\varepsilon_0\underline{\underline{\chi}}^{(1)}(\omega)\vec{e}E(\omega)$$

$$\vec{P}(t)=\vec{P}(\omega)e^{-i\omega t}+CC \qquad \vec{P}_{NL}(\omega)=3\varepsilon_0\underline{\underline{\underline{\chi}}}^{(3)}(\omega,-\omega,\omega)\vec{e}\vec{e}\vec{e}|E(\omega)|^2E(\omega)$$

Linear + Nonlinear polarization:

$$\vec{P}(\omega)=\varepsilon_0\left[\underline{\underline{\chi}}^{(1)}(\omega)\vec{e}+3\underline{\underline{\underline{\chi}}}^{(3)}(\omega,-\omega,\omega)\vec{e}\vec{e}\vec{e}|E(\omega)|^2\right]E(\omega)$$

$$\vec{P}(\omega)=\varepsilon_0\left[\chi_{eff}^{(1)}+\underbrace{\chi_{eff}^{(3)}|A(\omega)|^2}\right]A(\omega)e^{ikz}$$

Real part : $\chi_{eff}^{(3)'}$ induces a modification of the refractive index

$$n^2(\omega)=\underbrace{1+\chi_{eff}^{(1)}}_{n_0^2(\omega)}+\chi_{eff}^{(3)'}|A(\omega)|^2$$

$$n(\omega)=n_0(\omega)\sqrt{1+\underbrace{\frac{\chi_{eff}^{(3)'}}{2n_0^3\varepsilon_0c}I(\omega)}_{\varepsilon<<1}}\approx n_0(\omega)+n_2I(\omega)$$

Imaginary part : $\chi_{eff}^{(3)''}$ induces a modification of the absorption

I- Optical Kerr Effect

- ***Nonlinear refractive index***

$$n\approx n_0+n_2I$$

$$n_2=\frac{\chi_{eff}^{(3)}}{4n_0^2\varepsilon_0c}$$

Example : Pure Silica n_0=1,45 and n_2= 3 10^{-20} m^2/W

- ***Nonlinear Phase***

Temporal propagation

$$\phi_{NL}(t)=2\pi\frac{n_2I(t)\,L}{\lambda}$$

$$\Delta\omega=-\frac{d\phi_{NL}}{dt}=-2\pi\frac{n_2\,L}{\lambda}\frac{dI(t)}{dt}$$

Consequences : time-varying phase, Self-phase modulation, temporal soliton

Spatial propagation

$$\phi_{NL}(\vec{r})=2\pi\frac{n_2I(\vec{r})\,L}{\lambda}$$

Consequences : Self-focusing or self-defocusing effect, filamentation, spatial soliton

I- Optical Kerr Effect

- ## *Self-Phase Modulation (SPM) effect*

Nonlinear wave equation, assuming an instantaneous response of the material and neglecting linear distortion effects (dispersion and diffraction)

$$\frac{\partial A(\rho,z)}{\partial z} = \imath k_0 n_2 I(\rho,z) A(\rho,z) \quad \begin{cases} \rho = t & \text{Time} \\ & \text{or} \\ \rho = \boldsymbol{r} & \text{Space} \end{cases}$$

Lossless medium ➡ $I(\rho,z)$ = Cste at a fixed *t* or ***r***

Solution ➡

$$\begin{aligned} A(\rho,z) &= A(\rho,0)e^{\imath k_0 n_2 I(\rho) z} \\ &= A(\rho,0)e^{\imath \Phi_{\mathrm{NL}}(\rho,z)}, \end{aligned} \quad \begin{cases} |A(t,z)|^2 = |A(t,0)|^2 \\ \text{or} \\ |A(\mathbf{r},z)|^2 = |A(\mathbf{r},0)|^2 \end{cases}$$

⇒ ***The phase matching is automatically fulfilled***

⇒ *Assuming n2 is a purely real quantity,* ***the SPM induces a phase variation proportionally to the intensity***

⇒ ***The beam shape*** *(in time or space)* ***in intensity is conserved along the beam propagation*** *(neglecting linear distortion effects)*

I- Optical Kerr Effect

- ## *Self-Phase Modulation (SPM) effect*

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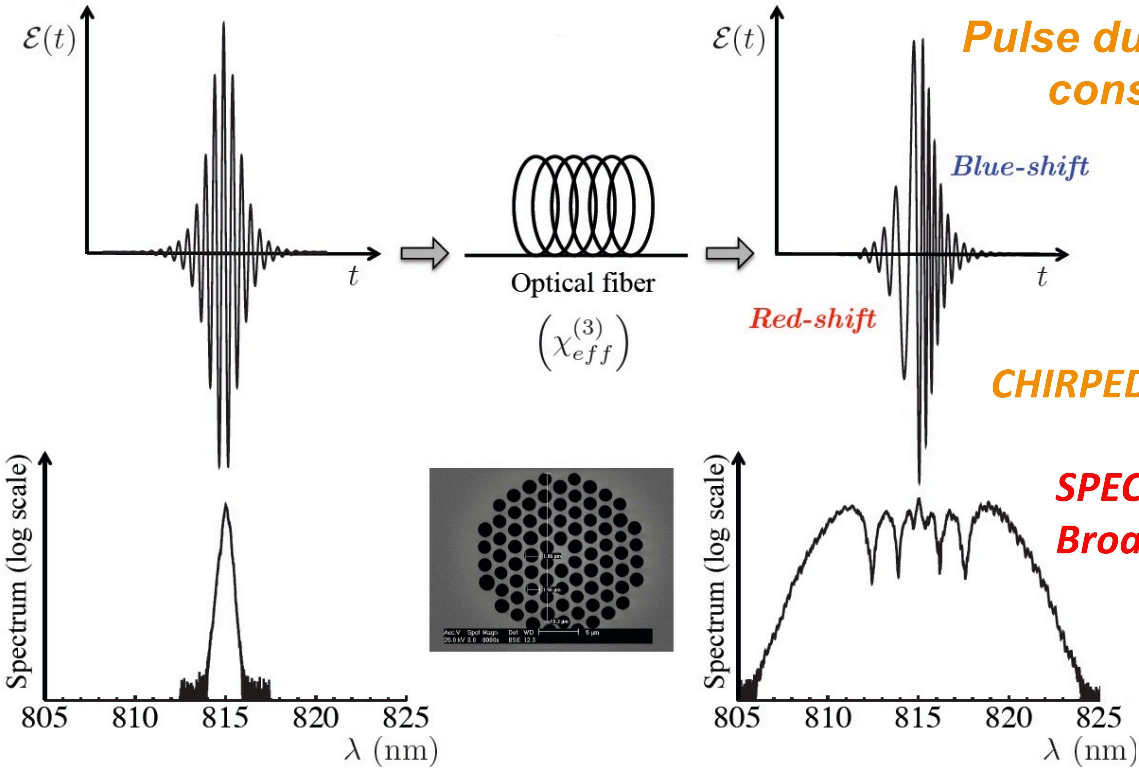
- Nonlinear phase of a transmitted pulse or beam $\quad \Phi_{\mathrm{NL}}(\rho,z) = k_0 n_2 I(\rho) z$
- Instantaneous frequency (case of a PULSE)

$$\Delta\omega_{\mathrm{NL}} \simeq -\frac{d\Phi_{\mathrm{NL}}(t)}{dt} = -k_0 n_2 \frac{dI(t)}{dt} z.$$

Spectral broadening + Frequency CHIRP of the pulse

Self-Phase Modulation



Pulse duration conserved

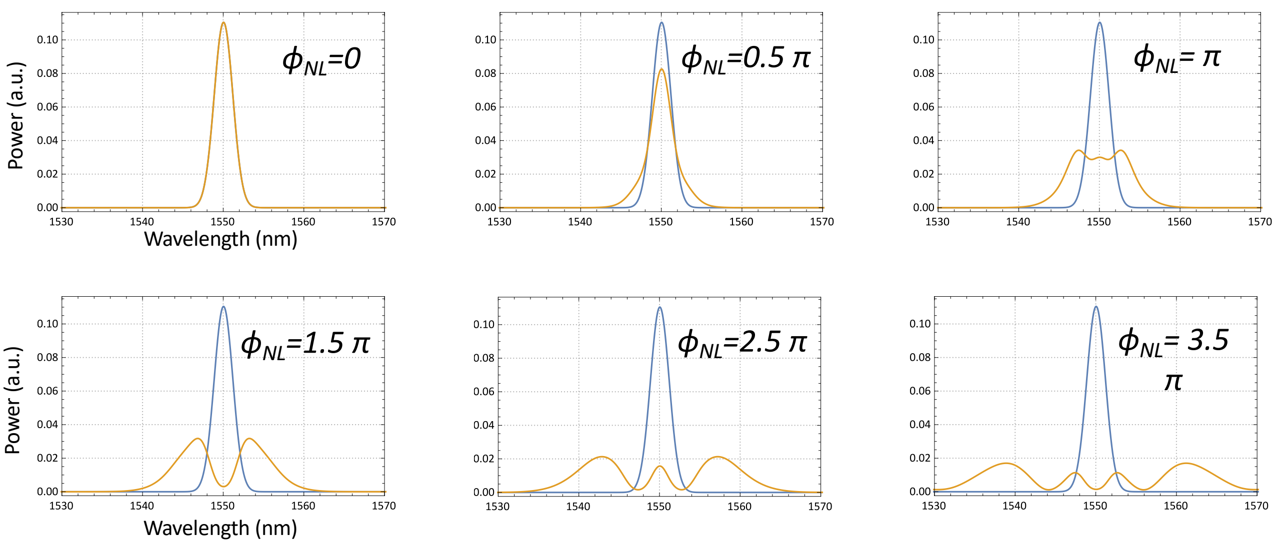
CHIRPED Pulse

SPECTRAL Broadening

From Margaux Barbier, LCF Manolia, PhD manuscript (2014)

Self-Phase Modulation

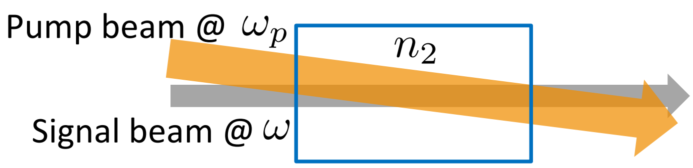
- *Spectral broadening through SPM – Gaussian Pulse*



I- Optical Kerr Effect

- ***Cross-Phase Modulation (XPM) effect***

The optical Kerr effect is then driven by the intensity of a pump wave @ω_P, leading to the modification of the phase experienced by a signal wave @ω



$$P^{(3)}(\omega = \omega_P - \omega_P + \omega)$$

$$P^{(3)}(\omega) = 6\epsilon_0 \chi^{(3)}_{eff} |A_P(\rho,z)|^2 A(\rho,z) \qquad \begin{cases} \rho = t & \text{Time} \\ & \text{or} \\ \rho = \boldsymbol{r} & \text{Space} \end{cases}$$

Nonlinear phase for the signal beam (pulse or beam) $\qquad \phi_{NL}(\rho,z)) = 2k_0 n_2 I_P(\rho) z$

$\Rightarrow$ ***In case of XPM, the nonlinear phase shift is x 2 compared to SPM*** *(consequence of the degeneracy factor)*

I- Optical Kerr Effect

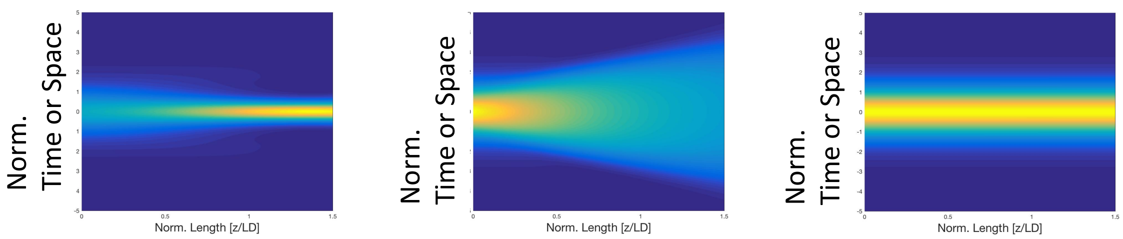
What have we learned ?

- propagation of a wave packet through a pure Kerr medium leads to a phase modification proportionally to the wave intensity.
- The phase matching condition is automatically fulfilled
- A pure Kerr and lossless medium is equivalent to a spatial or/and a temporal phase modulator (phase α Intensity) and leads to a spectral broadening

➜ ***SPATIAL / TEMPORAL KERR LENS EFFECT***

Question : interplays between linear and nonlinear Kerr effects?

Self-focusing & de-focusing effects | *Solion effect*



➜ ***Nonlinear Schrödinger Equation***

I- Optical Kerr Effect

- ***Nonlinear Schrödinger Equation – Time domain***

Propagation of a pulse : $$\begin{aligned}\boldsymbol{\mathcal{E}}(z,t) &= A(z,t)e^{\imath\beta_0 z}e^{-\imath\omega_0 t}\boldsymbol{e} + CC,\\ \boldsymbol{\mathcal{P}}_{NL}(z,t) &= \boldsymbol{\Pi}_{NL}(z,t)e^{\imath\beta_p z}e^{-\imath\omega_0 t} + CC.\end{aligned}$$

- We start with the wave equation in the frequency domain :

$$\nabla\times\nabla\times\boldsymbol{E}(\boldsymbol{r},\omega) = \frac{\omega^2}{c^2}\underline{\underline{\epsilon}}(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega) + \omega^2\mu_0\boldsymbol{P}^{(NL)}(\boldsymbol{r},\omega)$$

writing $$\begin{aligned}\boldsymbol{E}(z,\omega) &= \boldsymbol{e}\tilde{A}(z,\omega-\omega_0)e^{\imath\beta_0 z}\\ \boldsymbol{P}_{NL}(z,\omega) &= \tilde{\Pi}_{NL}(z,\omega-\omega_0)e^{\imath\beta_p z}\end{aligned}$$

$$\frac{\partial^2\tilde{A}(z,\omega-\omega_0)}{\partial z^2} + 2\imath\beta_0\frac{\partial\tilde{A}(z,\omega-\omega_0)}{\partial z} + \left[\beta^2(\omega)-\beta_0^2\right]\tilde{A}(z,\omega-\omega_0) = -\omega^2\mu_0\boldsymbol{e}\cdot\tilde{\Pi}_{NL}(z,\omega-\omega_0)e^{\imath\Delta\beta z}$$

I- Optical Kerr Effect

- ***Nonlinear Schrödinger Equation – Time domain***

- Power series expansion of $\beta(\omega)$: $\Delta\omega << \omega_0$

$$\begin{aligned}\beta(\omega) &= \beta_0 + \beta_1\Delta\omega + \frac{\beta_2}{2}\Delta\omega^2 + \cdots,\\ \beta^2(\omega) &\simeq \beta_0^2 + 2\beta_0\beta_1\Delta\omega + \left(\beta_1^2+\beta_0\beta_2\right)\Delta\omega^2 + \cdots\end{aligned}$$

- after substitution and following a Fourier transformation, the wave equation yields $\boldsymbol{\mathcal{E}}(t) = \int \boldsymbol{E}(\omega)e^{-\imath\omega t}d\omega$

$$\frac{\partial^2 A(z,t)}{\partial z^2} + 2\imath\beta_0\frac{\partial A(z,t)}{\partial z} + 2\imath\beta_0\beta_1\frac{\partial A(z,t)}{\partial t} - \left[\beta_1^2+\beta_0\beta_2\right]\frac{\partial^2 A(z,t)}{\partial t^2} = \mu_0\boldsymbol{e}\cdot\left[-\omega_0^2\boldsymbol{\Pi}_{NL}(z,t) - 2\imath\omega_0\frac{\partial\boldsymbol{\Pi}_{NL}(z,t)}{\partial t} + \frac{\partial^2\boldsymbol{\Pi}_{NL}(z,t)}{\partial t^2}\right]e^{\imath\Delta\beta z}$$

- Time coordinate transformation (introduction of a retarded time) : $\tau = t - z/v_g$ $\quad v_g = 1/\beta_1$ (Group velocity)

$$\frac{\partial^2 A(z,\tau)}{\partial z^2} - 2\beta_1\frac{\partial^2 A(z,\tau)}{\partial z\partial\tau} + 2\imath\beta_0\frac{\partial A(z,\tau)}{\partial z} - \beta_0\beta_2\frac{\partial^2 A(z,\tau)}{\partial\tau^2} = \mu_0\boldsymbol{e}\cdot\left[-\omega_0^2\boldsymbol{\Pi}_{NL}(z,\tau) - 2\imath\omega_0\frac{\partial\boldsymbol{\Pi}_{NL}(z,\tau)}{\partial\tau} + \frac{\partial^2\boldsymbol{\Pi}_{NL}(z,\tau)}{\partial\tau^2}\right]e^{\imath\Delta\beta z}$$

I- Optical Kerr Effect

- ***Nonlinear Schrödinger Equation – Time domain***

$$\frac{\partial^2 A(z,\tau)}{\partial z^2} - 2\beta_1 \frac{\partial^2 A(z,\tau)}{\partial z \partial \tau} + 2\imath\beta_0 \frac{\partial A(z,\tau)}{\partial z} - \beta_0\beta_2 \frac{\partial^2 A(z,\tau)}{\partial \tau^2} =$$
$$\mu_0 \boldsymbol{e} \cdot \left[-\omega_0^2 \boldsymbol{\Pi}_{NL}(z,\tau) - 2\imath\omega_0 \frac{\partial \boldsymbol{\Pi}_{NL}(z,\tau)}{\partial \tau} + \frac{\partial^2 \boldsymbol{\Pi}_{NL}(z,\tau)}{\partial \tau^2} \right] e^{\imath \Delta \beta z}$$

- Narrow spectral linewidth : $\Delta\omega << \omega_0$

$$\left|\frac{\partial A(z,t)}{\partial \tau}\right| \ll \omega_0 A(z,\tau) \quad \& \quad \left|2\beta_1 \frac{\partial^2 A(z,\tau)}{\partial z \partial \tau}\right| \ll 2\left|\beta_0 \frac{\partial A(z,\tau)}{\partial z}\right|$$

- Slow variation of $P_{NL}(t)$ within a time period $T=2\pi/\omega_0$

$$\omega_0^2 \boldsymbol{\Pi}_{NL}(z,\tau) \gg 2\omega_0 \frac{\partial \boldsymbol{\Pi}_{NL}(z,\tau)}{\partial \tau} \gg \frac{\partial^2 \boldsymbol{\Pi}_{NL}(z,\tau)}{\partial \tau^2}$$

- Slowly-varying amplitude approximation $\left|\frac{\partial^2 A(z,\tau)}{\partial z^2}\right| \ll \left|\beta_0 \frac{\partial A(z,\tau)}{\partial z}\right|$

$$\boxed{\frac{\partial A(z,\tau)}{\partial z} + \frac{\imath\beta_2}{2} \frac{\partial^2 A(z,\tau)}{\partial \tau^2} = \frac{\imath\omega}{2\epsilon_0 n c} \boldsymbol{e} \cdot \boldsymbol{\Pi}_{NL}(z,\tau) e^{\imath \Delta k z}}$$

I- Optical Kerr Effect

- ***Nonlinear Schrödinger Equation – TEMPORAL domain***

In presence of Kerr effect

$$\boldsymbol{\Pi}_{NL}(t,z) = \epsilon_0 \chi_{\text{eff}}^{(3)} |A(t,z)|^2 A(t,z) \boldsymbol{e}$$

Assumption : instantaneous response of Kerr effect $\gamma' = \frac{\omega_0}{2nc}\chi_{\text{eff}}^{(3)}$

NONLINEAR SCHRÖDINGER EQUATION

$$\boxed{\frac{\partial A(\tau,z)}{\partial z} + \frac{\imath\beta_2}{2} \frac{\partial^2 A(\tau,z)}{\partial \tau^2} - \imath\gamma' |A(\tau,z)|^2 A(\tau,z) = 0}$$

2rd order dispersion effect

Nonlinear Kerr term (phase variation α Intensity)

I- Optical Kerr Effect

- ***Nonlinear Schrödinger Equation – <u>SPATIAL</u> domain***

In presence of Kerr effect

$$\mathbf{\Pi}_{NL}(\boldsymbol{r},z) = \epsilon_0 \chi^{(3)}_{\text{eff}} |A(\boldsymbol{r},z)|^2 A(\boldsymbol{r},z)\boldsymbol{e}$$

$$\gamma' = \frac{\omega_0}{2nc}\chi^{(3)}_{\text{eff}}$$

NONLINEAR SCHRÖDINGER EQUATION

$$\boxed{\frac{\partial A(\boldsymbol{r},z)}{\partial z} + \frac{1}{2\imath k}\Delta_T A(\boldsymbol{r},z) - \imath\gamma |A(\boldsymbol{r},z)|^2 A(\boldsymbol{r},z) \quad = \quad 0}$$

Diffraction effect

Nonlinear Kerr term (phase variation α Intensity)

I- Optical Kerr Effect

- ***Dispersion length & Nonlinear length***

Variables transformation $T = \frac{\tau}{\tau_0}$ $\quad A(t,z) = \sqrt{\frac{I_0}{2nc\epsilon_0}}u(t,z)$

$$\frac{\partial u}{\partial z} + \frac{\imath sign(\beta_2)}{2L_D}\frac{\partial^2 u}{\partial T^2} - \imath\frac{|u|^2 u}{L_{NL}} = 0$$

with τ_0 : pulse duration

P_0 : peak power of the pulse

$L_D = \frac{\tau_0^2}{|\beta_2|}$ Dispersion length

$L_{NL} = \frac{1}{k_0 n_2 I_0}$ Nonlinear length

Once the interaction length is set :

$L << L_D$ The dispersive effect can be neglected

$L << L_{NL}$ The optical Kerr effect can be neglected

I- Optical Kerr Effect

- ***Nonlinear Schrödinger Equation***
  - **NLSE : physics insights**

$$\frac{\partial u}{\partial z} + \frac{\imath sign(\beta_2)}{2L_D}\frac{\partial^2 u}{\partial T^2} - \imath\frac{|u|^2 u}{L_{NL}} = 0$$

Dispersion effect

$$\frac{\partial u}{\partial z} + \frac{\imath sign(\beta_2)}{2L_D}\frac{\partial^2 u}{\partial T^2} - \imath\frac{|u|^2 u}{L_{NL}} = 0$$

Kerr effect

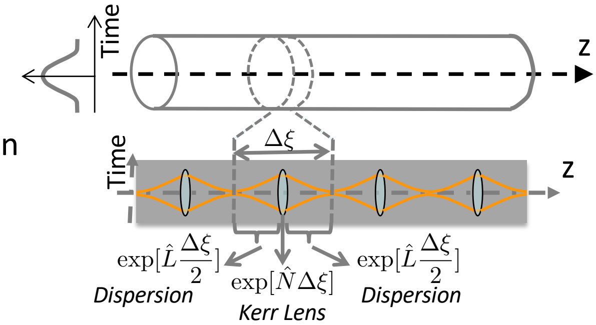
$$u(T,z) = u(T,0)e^{\imath\frac{|u(T)|^2}{L_{NL}}z}$$

Kerr LENS

APPLICATION :

$$\frac{\partial u}{\partial \xi} + \hat{L}u + \hat{N}u = 0$$

Numerical simulation « Spilt Step Fourier Method »



I- Optical Kerr Effect

Nonlinear Schrödinger Equation

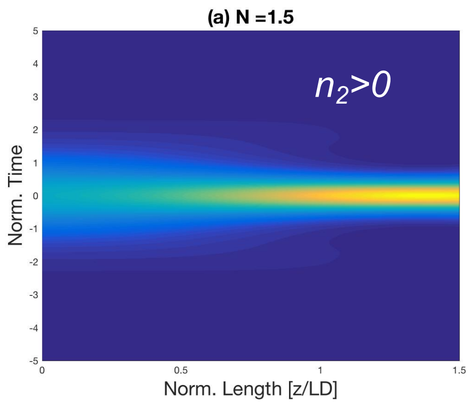
Variables transformation $\xi = z/L_D$ $N^2 = L_D/L_{NL}$

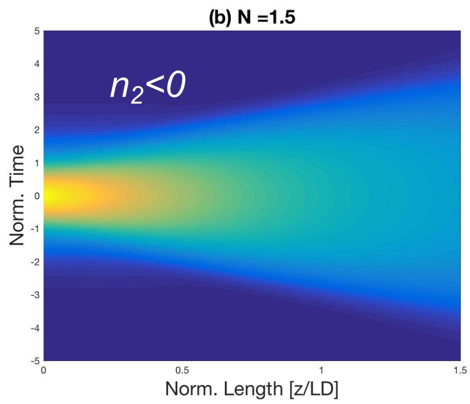
$$\boxed{\imath\frac{\partial u}{\partial \xi} - sign(\beta_2)\frac{1}{2}\frac{\partial^2 u}{\partial T^2} + N^2|u|^2 u = 0}$$

- ***APPLICATIONS : Self-focusing & de-focusing effects*** $\boxed{N > 1}$

Anomalous dispersion regime

$\beta_2 < 0$





I- Optical Kerr Effect

Nonlinear Schrödinger Equation

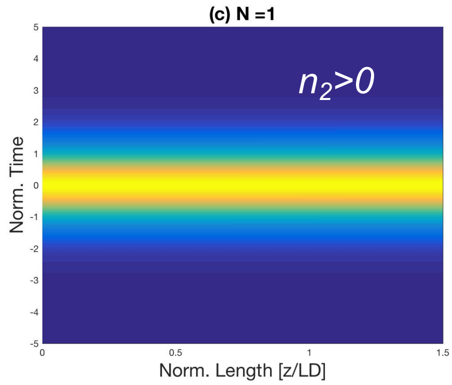
Variables transformation $\xi = z/L_D$ $\qquad N^2 = L_D/L_{NL}$

$$\boxed{\imath\frac{\partial u}{\partial \xi} - sign(\beta_2)\frac{1}{2}\frac{\partial^2 u}{\partial T^2} + N^2|u|^2 u = 0}$$

- ***APPLICATIONS : Solion effect***

Anomalous dispersion regime

$\beta_2 < 0$



$\boxed{N = 1}$

Strict compensation of the dispersive effect by the optical Kerr effect

$\Rightarrow$ Propagation of the pulse without deformation

$\Rightarrow$ Soliton solutions :

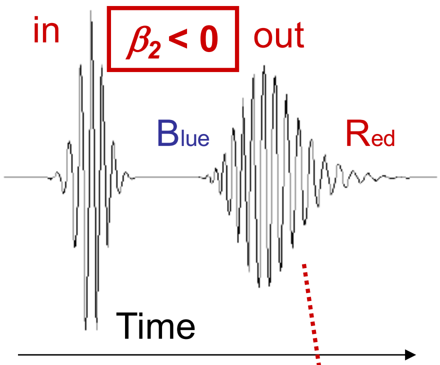
$$u(\rho) = N\mathrm{sech}(\rho)$$

For further reading : Nonlinear Fiber Optics, Ch.5 by Govind P. Agrawal

I- Optical Kerr Effect

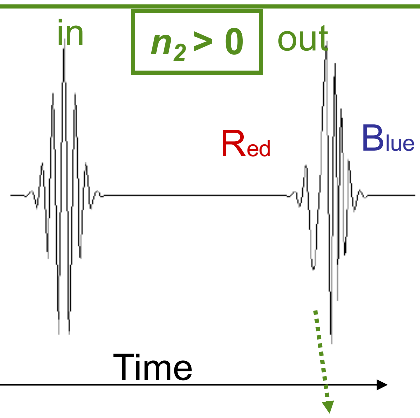
- ***Pulse propagation : influence of the DISPERSION EFFECT and the OPTICAL KERR EFFECT n_2 ...***

Pulse propagation in a dispersive medium (anomalous dispersion, D>0)



Pulse spreading

Pulse propagation in a NL medium (n_2 > 0)



NO pulse spreading
BUT intro . *''CHIRP''*

Schéma pulse copié de http://www.sccs.swarthmore.edu/users/02/lisal/physics/presentations/soliton.pdf

I- Optical Kerr Effect

- ***Temporal Soliton***

Pulse propagation in a NL medium ($n_2 > 0$) and dispersive ($\beta_2 < 0$)

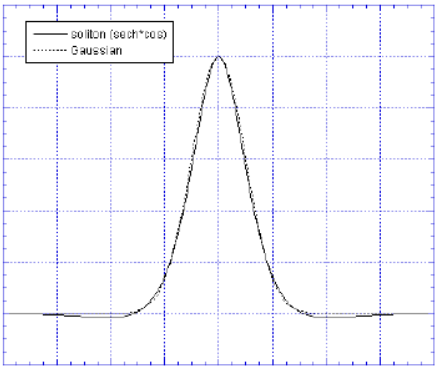
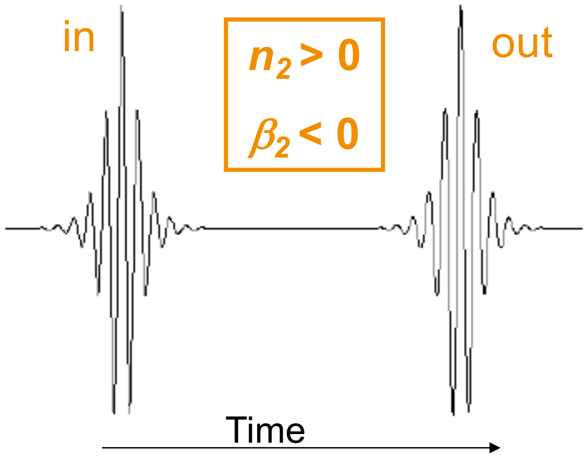


Figure 3: Envelope of soliton solution to NLSE compar envelope of a Gaussian pulse.



Soliton solution = « *sech* » pulse shape with no deformation

Full compensation between group-velocity dispersion and the effect of self-phase modulation

http://www.sccs.swarthmore.edu/users/02/lisal/physics/presentations/soliton.pdf

I- Optical Kerr Effect

- ***NL Schrödinger Eq. :***

temporal Soliton

- fundamental Soliton N=1 :

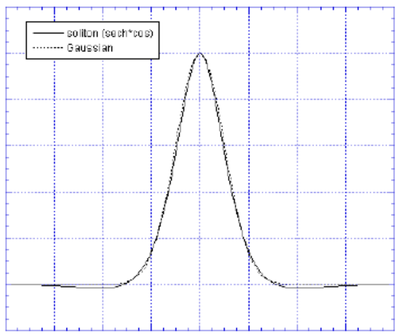
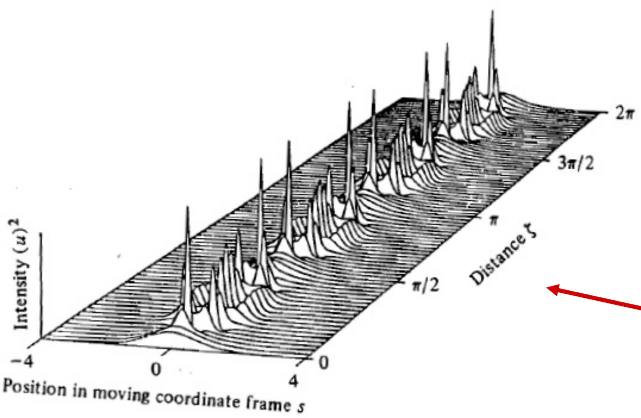


Figure 3: Envelope of soliton solution to NLSE compared with envelope of a Gaussian pulse.



- Soliton N=4 : shape

Fig. 7.8 Computer-generated solutions of the nonlinear Schrödinger equation showing the evolution of an $N = 4$ soliton as it propagates over a distance equal to four soliton periods. (After Blow and Doran, 1987.)

Fig. extraite du livre de Butcher & Cotter « The elements of nonlinear optics »