NONLINEAR OPTICS - Tutorial
Self-Phase Modulation effects and nonlinear Schrödinger equation

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1 Nonlinear Schrödinger Equation

Propagation of an optical pulse through a nonlinear Kerr material (with an instantaneous response) is described by the nonlinear Schrödinger equation (refer to lecture notes for demonstration):

$$\frac{\partial A(\tau,z)}{\partial z} + \frac{\imath\beta_2}{2}\frac{\partial^2 A(\tau,z)}{\partial \tau^2} - \imath\gamma|A(\tau,z)|^2 A(\tau,z) = 0, \tag{1}$$

with $\gamma = \frac{\omega_0}{2nc}\chi^{(3)}_{\text{eff}}$.

1. What do the last two terms in the equation represent?
2. The following new variables are introduced:
   - $T = \frac{\tau}{\tau_0}$, with τ_0 the pulse duration,
   - $A(\tau,z) = \sqrt{\frac{I_0}{2nc\epsilon_0}}u(\tau,z)$, with I_0 the pulse peak intensity.

   Show that the nonlinear wave equation yields:

   $$\frac{\partial u}{\partial z} + \frac{\imath sign(\beta_2)}{2L_D}\frac{\partial^2 u}{\partial T^2} - \imath\frac{|u|^2 u}{L_{NL}} = 0 \tag{2}$$

   What are the expressions for the quantities L_D and L_{NL}? Check their homogeneity.
3. Under which condition(s) on the propagation length, the effects described in equation (2) will have to be taken into account?
4. Solve equation (2) in the situation: $L_D \gg L_{NL}$, considering a lossless material. After a distance z, show that the pulse have accumulated a nonlinear phase $\phi_{NL} = k_0 n_2 I_0 z$, with n_2 the nonlinear refractive index of the material.

2 Self-Phase Modulation effect in an AlGaAs integrated waveguide

We next consider pulse propagation through an AlGaAs integrated waveguide. The waveguide operates in single mode regime at 1550 nm, with an effective mode area $A_{eff} = 10^{-9}\,\text{cm}^2$ and a length $L = 1\,\text{mm}$. We study the consequence of optical Kerr effect on gaussian pulses, with a duration $\tau_0 = 1.6\,\text{ps}$. At this operating wavelength, the material can be considered as lossless and its nonlinear refractive is taken equal to $n_2 = 3.8\,10^{-13}\,\text{cm}^2/\text{W}$.

1. The second order dispersion coefficient of the waveguide has been calculated by simulation and is estimated equal to $\beta_2 = -6.4\,\text{ps}^2/\text{cm}$. Calculate the corresponding dispersion length L_D. Conclusion.
2. Calculate the peak power required for the pulse to accumulate a nonlinear phase shift $\phi_{NL} = \pi$.

3. Calculate the corresponding value for L_{NL} and compare with L and L_D values. Conclusion.
4. The figure below shows optical spectra measured at the waveguide output for injected powers respectively equal to (a) 0.13 W, (b) 1.02 W and (c) 2.05 W. The measured output spectra are plotted with black curves, the simulated spectra with grey curves. The dashed lines show the pulse spectrum at low power. Explain the shapes of spectra (a) and (c).

   For a gaussian pulse, the relation between the FWHM for the duration and the spectral linewidth are given by the relation: $\Delta\tau.\Delta\nu = 0.886$. For the spectrum shown in (c), the pulse shape will be approximated by a triangular shape.

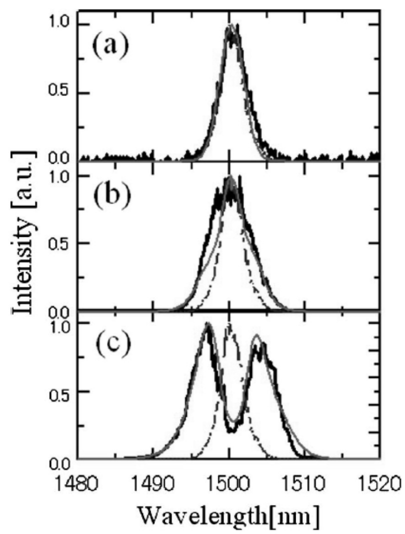

FIG. 1 – *Output spectra measured at the waveguide output for injected powers respectively equal to (a) 0.13 W, (b) 1.02 W and (c) 2.05 W.*

3 Numerical simulation

Hereafter, we intend to implement the numerical simulation of the nonlinear wave equation (2), *Non-Linear Schrödinger Equation* (NLSE). The objective of this section is to solve the NLSE using the Split Step Fourier method, which is described in Appendix.

1. Neglecting the dispersion effect, calculate the spectra of the output pulse with the parameters used in the previous section. Plot these spectra and compare with the measured spectra reported in the figure 1.
2. Implement the dispersive term in the NLSE. Verify that the numerical solutions are consistent with the expected behaviors (for instance by switching off the nonlinear term, and by varying either the pulse duration or the waveguide length).
3. By varying the parameters in the NLSE equation, illustrate the self-focusing, the de-focusing, and the soliton effects.

APPENDIX : Numerical simulation: the Split Step Fourier method

The wave equation (2) can be written in a following symbolic form:

$$\frac{\partial u}{\partial \xi} + \hat{L}u + \hat{N}u = 0, \tag{3}$$

with $\hat{L} = +\imath\frac{sign(\beta_2)}{2}\frac{\partial^2}{\partial T^2}$, a linear operator that accounts for the second-order dispersion, and $\hat{N} = -\imath N^2|u|^2$ a nonlinear operator. A general solution takes the form:

$$u(\rho,\xi + \Delta\xi) = \exp\left((\hat{L} + \hat{N})\Delta\xi\right)u(\rho,\xi).$$

The operator $\exp\left((\hat{L} + \hat{N})\Delta\xi\right)$ takes into account simultaneously the linear and nonlinear effects. However, one can simplify this operator using the Baker-Hausdorff formula [1]:

$$\exp\left((\hat{L} + \hat{N})\Delta\xi\right) \simeq \exp\left(\hat{N}\Delta\xi\right)\exp\left(\hat{L}\Delta\xi\right)\exp\left(-[\hat{L},\hat{N}](\Delta\xi)^2\right) \tag{4}$$

The exponential term $[\hat{L},\hat{N}] = \hat{L}\hat{N} - \hat{N}\hat{L}$ can be neglected as it contains a second order term in $(\Delta\xi)^2$. The wave equation (3) can then be approximated by the relation:

$$u(\rho,\xi + \Delta\xi) \simeq \exp[\hat{L}\frac{\Delta\xi}{2}]\exp[\hat{N}\Delta\xi]\exp[\hat{L}\frac{\Delta\xi}{2}]u(\rho,\xi). \tag{5}$$

Whereas the propagation is governed by two simultaneous effects, the linear and optical Kerr effects, equation (5) approximates the propagation by separating these two contributions. Such an approximation is valid since the propagation distance $\Delta\xi$ is kept small. The medium is then split along ξ in slices with an equal thickness $\Delta\xi$. The numerical resolution follows the iterative algorithm describes in Fig. (2). Within each slice of medium, one first apply the linear term (diffraction or dispersion) under which the wave packet undergoes an expansion (in space or time). In general, this calculation step is achieved in the Fourier domain as the linear operator in the Fourier domain takes the form of a phasor. Following this first step, the nonlinear operator is applied, symbolically described by a thin lens effect in Fig. (2). One reminds that the Kerr lens effect can be either a convergent or divergent lens depending on the sign of the nonlinear refractive index n_2.

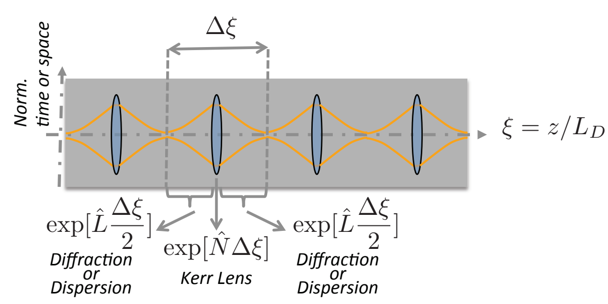

FIG. 2 – *Split-Step Fourier algorithm for a numerical simulation of the NLSE that described the nonlinear propagation a wave packet (pulse or beam) through a nonlinear medium along the direction ξ. The medium is divided in cells of thickness $\Delta\xi \ll 1$. Within each cell, one successively apply the linear and nonlinear operators.*

1. Approximation used in quantum physics to simplify quantum operators. For instance, refer to chapter 4 of the book "Contemporary optical image processing with MathLab" from Poon and Banerjee, Elsevier (2001).