Optimisation Discrète Non linéaire
M1 Informatique site d'Orsay

Integer Non Linear Programming:

- 0-1 Quadratic Programming
- Pure Integer separable quadratic programming
- Pure Integer nonseparable quadratic programming

What is Quadratic terms ?

When you need to choose one decision variable and an other decision variable, at the same time.

When the problem becomes quadratic ?

(QKP) : maximiser $f(x) = \sum b_i x_i + \sum\sum g_{ij} x_i x_j$

sous les contraintes :

$\sum w_i x_i \leq W$

$x_i \in \{0;1\}$

QKP

First remark : (QKP) not always convex, it depends on the nature of the matrix Q

- if Q is SDP => (QKP) convex
- If so => a solver like CPLEX can solve directly the problem

2nd remark : linear solver are well known and used, so it is usual to find/search linearization techniques to deal with a linear problem (equivalent) instead of non linear initial problem

Linearization of the quadratic term

For all $x \in \{0;1\}$, for all $y \in [0, U(y)]$, and for all $e \in R$, $e=xy$ if and only if the following constraints are satisfied :

$e <= x\, U(y)$

$e <= y$

$e >= y - (1-x)U(y)$

$e >= 0$

Linearization of the quadratic term

Proof :

Assuming that e=xy. Let us prove that =>

Let examine the two possible values of the variable x.

→ if x=0 then e=0 and the previous system of inequations becomes :

0 <=0

0<=y

0>= y-U(y)

0>=0

System 1 satisfied !

Linearization of the quadratic term

<u>Proof :</u>

→ if x=1 then the inequations system becomes :

e <=U(y)

e<=y

e >= y

e >=0

Then we have that e = y !

Linearization of the quadratic term

Proof :

Let us now assume that the system of inequations is satisfied and let us prove that <=

Let us examine the two possible values of the variable x.

→ if x=0 then the system becomes

$e <= 0$

$e <= y$

$e >= y-U(y)$

$e >= 0$

then : $e = 0$

An exemple : linaarization

Replacing the product x1x4 by new variable e14 and the product x2x3 by e23.

Adding constraints linearization to force variable e14 to take the value of the product x1x4 and variable e23 to take the value of the product x2x3.

An example

We obtain the following mixed integer linear program :

(LP1) : maximize f(x)= 2 x1 + 5 x2 – x4 + 2e14

Subjected to :

x2 + x3 – 3 x4 + 3 e23 <= 2 e14

x1 – x2 + 2 x3 + 2 e23 >= 12

e14 <= 8 x1 ;

e14 <= x4

e14 >= x4 – 8(1-x1)

e14 >=0

e23 <= 5 x2

e23 <= x3

e23 >= x3 – 5(1-x2)

e23 >=0

x1, x2 ∈ {0;1} ; 0 <= x3 <= 5 ; 0 <= x4 <= 8

An optimal solution is the following

x1=1, x2=1,x3=3,x4=8,e14=8 et e23=3.

The objective value is equal to 15.

An optimal solution of QP1 is then
x1=1, x2=1,x3=3,x4=8

Extension to the cross term when the two variables are 0-1

xy can be replaced by a 0-1 variable denoted by e (for instance) and we have to add the following constraints :

$e \le x$

$e \le y$

$1 - x - y + e \ge 0$

$e \ge 0$